

THE ONE-DIMENSIONAL HEAT EQUATION IN THE ALEXIEWICZ NORM

ERIK TALVILA

ABSTRACT. A distribution on the real line has a continuous primitive integral if it is the distributional derivative of a function that is continuous on the extended real line. The space of distributions integrable in this sense is a Banach space that includes all functions integrable in the Lebesgue and Henstock–Kurzweil senses. The one-dimensional heat equation is considered with initial data that is integrable in the sense of the continuous primitive integral. Let $\Theta_t(x) = \exp(-x^2/(4t))/\sqrt{4\pi t}$ be the heat kernel. With initial data f that is the distributional derivative of a continuous function, it is shown that $u_t(x) := u(x, t) := f * \Theta_t(x)$ is a classical solution of the heat equation $u_{11} = u_2$. The estimate $\|f * \Theta_t\|_\infty \leq \|f\|/\sqrt{\pi t}$ holds. The Alexiewicz norm is $\|f\| = \sup_I |\int_I f|$, the supremum taken over all intervals. The initial data is taken on in the Alexiewicz norm, $\|u_t - f\| \rightarrow 0$ as $t \rightarrow 0^+$. The solution of the heat equation is unique under the assumptions that $\|u_t\|$ is bounded and $u_t \rightarrow f$ in the Alexiewicz norm for some integrable f . The heat equation is also considered with initial data that is the n th derivative of a continuous function and in weighted spaces such that $\int_{-\infty}^\infty f(x) \exp(-ax^2) dx$ exists for some $a > 0$. Similar results are obtained.

1. INTRODUCTION

The one-dimensional heat equation is the canonical parabolic partial differential equation of second order. For simple geometries solutions can be represented explicitly as series or integrals. It is well-known that with heat conduction on an infinite rod the solution is given by a convolution of initial data with the heat kernel. The classical problem is:

$$(1.1) \quad u \in C^2(\mathbb{R}) \times C^1((0, \infty)) \text{ such that}$$

$$(1.2) \quad u_t - u_{xx} = 0 \text{ for } (x, t) \in \mathbb{R} \times (0, \infty).$$

The Gauss–Weierstrass heat kernel is $\Theta_t(x) = \Theta(x, t) = (4\pi|t|)^{-1/2}e^{-x^2/(4t)}$ (defined for $t \neq 0$). The solution of (1.1)–(1.2) is then given by the convolution of

Date: Preprint January 16, 2015. To appear in *Advances in Pure and Applied Mathematics*. 1991 *Mathematics Subject Classification*. Primary 35K05, 46F05. Secondary 26A39, 46F10, 46G12.

Key words and phrases. heat equation, continuous primitive integral, Henstock–Kurzweil integral, Schwartz distribution, generalised function, Alexiewicz norm, partial differential equation, convolution, Banach space.

Written while visiting the Department of Mathematics, University of Arizona. Thank you for the generous hospitality.

initial data f with the heat kernel

$$(1.3) \quad f * \Theta_t(x) = \int_{-\infty}^{\infty} f(x - \xi) \Theta_t(\xi) d\xi = \int_{-\infty}^{\infty} f(\xi) \Theta_t(x - \xi) d\xi.$$

There are various ways to impose initial conditions. If f is bounded and continuous on \mathbb{R} then $f * \Theta_t(x)$ is continuous on $\mathbb{R} \times [0, \infty)$ and $\|f * \Theta_t - f\|_{\infty} \rightarrow 0$ as $t \rightarrow 0^+$. If $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$ then $\|f * \Theta_t - f\|_p \rightarrow 0$ as $t \rightarrow 0^+$. For example, see [5], [29], [14].

In this paper we allow the integral in (1.3) to exist as a continuous primitive integral. This integration process includes the Lebesgue and Henstock–Kurzweil integrals with respect to Lebesgue measure. An attractive feature of this integral is that the space of integrable distributions is a Banach space isometrically isomorphic to the continuous functions on the extended real line that vanish at $-\infty$. These distributions tend to behave much more like L^p functions than arbitrary distributions and many properties typical of functions hold in these spaces, such as integration by parts, continuity in norm, Hölder inequality, etc. Whereas results involving arbitrary distributions hold weakly, here we have results holding in the strong (norm) sense.

The main idea behind the continuous primitive integral is to define a class of continuous functions (the primitives) that is a Banach space under a uniform or weighted uniform norm. The integrable distributions are then defined to be the distributional derivatives of the primitives. Integration is defined via the fundamental theorem of calculus: If F is a continuous function on $\overline{\mathbb{R}}$ then its distributional derivative F' is integrable and $\int_a^b F' = F(b) - F(a)$ for all $a, b \in \overline{\mathbb{R}}$. The space of integrable distributions is a Banach space isometrically isomorphic to the space of primitives. See the following section for more detail.

We define three classes of integrable distributions. First, primitives are the continuous functions on the extended real line, vanishing at $-\infty$ (Section 3). This integral then includes the Lebesgue and Henstock–Kurzweil integrals with respect to Lebesgue measure. The formula $\int_a^b F' = F(b) - F(a)$ continues to hold when F is continuous, even if the pointwise derivative vanishes almost everywhere or fails to exist at any point. Secondly, we take higher derivatives of such primitives (Section 4). This includes functions with algebraic singularities of order $(x - x_0)^{-a}$ at any point $x_0 \in \mathbb{R}$, for all exponents $a > 0$. Finally, we use primitives such that the weighted integral $\int_{-\infty}^{\infty} f(x) \exp(-ax^2) dx$ exists for some $a > 0$ (Section 5). This includes the L^p spaces with respect to Lebesgue measure and weights $\exp(-ax^2)$ ($a > 0$).

For each of these cases we prove theorems of the following type, the respective norm shown generically as $\|\cdot\|$. For f integrable in one of the three above senses: $f * \Theta_t(x)$ is separately analytic in x and t ; is a classical solution of the heat equation (1.1)-(1.2); satisfies an inequality $\|f * \Theta_t\| \leq C_t \|f\|$ for a sharp constant C_t . The initial data is taken on in the sense that if $u_t(x) = f * \Theta_t(x)$ then

$$(1.4) \quad \|u_t - f\| \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

A uniqueness theorem is proved under (1.1)-(1.2) with the only additional assumptions $\|u_t\|$ is bounded and $\|u_t - f\| \rightarrow 0$ as $t \rightarrow 0^+$ for some integrable

distribution f (Section 6). Pointwise and norm estimates are given for $f * \Theta_t$ and its derivatives.

An extensive bibliography on classical results appears in [5]. See also [29]. For initial data in L^p spaces see [13] and [14]. The heat equation is considered in weighted spaces of signed measures in [27]. Distributional solutions are studied in [6] and [20]. These last two works consider tempered distributions, for which Fourier transform methods apply. The initial data is taken on in the distributional (weak) sense. An important feature of our results is that all of the distributions we consider (some tempered, some not tempered) are in Banach spaces and the initial data is taken on in the strong (norm) sense.

2. THE CONTINUOUS PRIMITIVE INTEGRAL

The notation we use for distributions is standard. The test functions are $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$. Distributions are denoted $\mathcal{D}'(\mathbb{R})$. We will usually denote distributional derivatives by F' and pointwise derivatives by $F'(t)$. If f is a locally integrable function in the Lebesgue or Henstock–Kurzweil sense we identify f with its distribution T_f defined by $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x) dx$ for $\phi \in \mathcal{D}(\mathbb{R})$. The results on distributions we use can be found in [9] or [8].

Denote the extended real line by $\overline{\mathbb{R}} = [-\infty, \infty]$. We define $C(\overline{\mathbb{R}})$ to be the continuous functions F such that $F(\infty) = \lim_{x \rightarrow \infty} F(x)$ and $F(-\infty) = \lim_{x \rightarrow -\infty} F(x)$ both exist as real numbers. The space of primitives for the continuous primitive integral is $\mathcal{B}_c = \{F \in C(\overline{\mathbb{R}}) \mid F(-\infty) = 0\}$. Note that since $F(-\infty) = 0$, \mathcal{B}_c is a Banach space under the norm $\|F\|'_\infty = \sup_{x < y} |F(x) - F(y)|$. Let $\mathcal{A}_c = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F' \text{ for some } F \in \mathcal{B}_c\}$. We have made the arbitrary but convenient choice of making primitives vanish at $-\infty$. A consequence is that if $f \in \mathcal{A}_c$ then it has a unique primitive $F \in \mathcal{B}_c$. Otherwise, the primitives of f would differ by a constant. The integral is then defined $\int_a^b f = F(b) - F(a)$ for all $a, b \in \overline{\mathbb{R}}$. Hence, a distribution f has a continuous primitive integral if it is the distributional derivative of a function $F \in \mathcal{B}_c$, i.e., for all $\phi \in \mathcal{D}(\mathbb{R})$ we have $\langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x) dx$. Since F and ϕ' are continuous and ϕ' has compact support, the integral defining the derivative exists as a Riemann integral. The Alexiewicz norm of $f \in \mathcal{A}_c$ is $\|f\| := \sup_{x < y} |\int_x^y f| = \sup_{x < y} |F(x) - F(y)| := \|F\|'_\infty$ where $F \in \mathcal{B}_c$ is the primitive of f . Note that \mathcal{A}_c is a Banach space isometrically isomorphic to \mathcal{B}_c . We get an equivalent norm when we fix $x = -\infty$. Then $\|f\|' := \|F\|_\infty$ and this makes \mathcal{A}_c into a Banach space that is isometrically isomorphic to \mathcal{B}_c with the uniform norm. For $F \in \mathcal{B}_c$, $\|F\|_\infty \leq \|F\|'_\infty \leq 2\|F\|_\infty$. However, the form of Alexiewicz norm we are using is somewhat more convenient. Note that L^1 and the spaces of Henstock–Kurzweil and wide Denjoy integrable functions are contained in \mathcal{A}_c since these spaces all have primitives that are continuous [10]. (They are in fact dense subspaces of \mathcal{A}_c .) The function $f(x) = x^{-2} \sin(x^{-3})$ is not in L^1_{loc} but is in \mathcal{A}_c , as can be seen via integration by parts. If $F \in C(\overline{\mathbb{R}})$ and singular ($F'(x) = 0$ for almost all $x \in \mathbb{R}$) then the Lebesgue integral of F' exists and is $\int_E F'(x) dx = 0$ for each measurable set E . But $F' \in \mathcal{A}_c$ with continuous

primitive integral $\int_a^b F' = F(b) - F(a)$. If $F \in C(\overline{\mathbb{R}})$ such that it has a pointwise derivative nowhere then the Lebesgue integral of F' is meaningless but $F' \in \mathcal{A}_c$ with continuous primitive integral $\int_a^b F' = F(b) - F(a)$. The Alexiewicz norm seems to first appear in [1]. The continuous primitive integral has its genesis in the work of Mikusiński and Ostaszewski [19]; Bongiorno and Panchapagesan [4]; Ang, Schmitt, Vy [2]; Bäumer, Lumer and Neubrander [3]. For a detailed overview, see [22].

If $g: \mathbb{R} \rightarrow \mathbb{R}$ its variation is $Vg = \sup \sum |g(x_i) - g(y_i)|$ where the supremum is taken over all disjoint intervals $(x_i, y_i) \subset \mathbb{R}$. The functions of bounded variation are denoted \mathcal{BV} . If $g \in \mathcal{BV}$ then it has limits at infinity and we define $g(\pm\infty) = \lim_{x \rightarrow \pm\infty} g(x)$. Functions of bounded variation form the multipliers for \mathcal{A}_c . If $f \in \mathcal{A}_c$ with primitive $F \in \mathcal{B}_c$ and $g \in \mathcal{BV}$ then the integration by parts formula is

$$(2.1) \quad \int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F(x) dg(x).$$

The last integral is a Henstock–Stieltjes integral. See, for example, [18]. When g is absolutely continuous the formula simplifies to the Lebesgue integral

$$(2.2) \quad \int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F(x)g'(x) dx.$$

A type of Hölder inequality for $f \in \mathcal{A}_c$ and $g \in \mathcal{BV}$ is [21, Lemma 24]

$$(2.3) \quad \left| \int_{-\infty}^{\infty} fg \right| \leq \left| \int_{-\infty}^{\infty} f \right| \inf |g| + \|f\| Vg \leq \|f\| (|g(\infty)| + Vg).$$

A convergence theorem for the continuous primitive integral [22, Theorem 22]:

Theorem 2.1. *Let $f \in \mathcal{A}_c$. Suppose $\{g_n\} \subset \mathcal{BV}$ such that there is $M \in \mathbb{R}$ so that for all $n \in \mathbb{N}$, $Vg_n \leq M$. If $g_n \rightarrow g$ on $\overline{\mathbb{R}}$ for a function $g \in \mathcal{BV}$ then $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} fg_n = \int_{-\infty}^{\infty} fg$.*

If $f \in \mathcal{A}_c$ and $g \in \mathcal{BV}$ then the convolution $f * g(x) = \int_{-\infty}^{\infty} f(x - \xi)g(\xi) d\xi$ is well-defined on \mathbb{R} . The convolution is continuous and $\|f * g\|_{\infty} \leq \|f\|(\|g\|_{\infty} + Vg)$. Properties of the convolution are proven for the continuous primitive integral in [23]. It is shown there by a limiting process that $f * g$ also exists for $g \in L^1$ and that $\|f * g\| \leq \|f\|\|g\|_1$.

Three facts about the heat kernel:

$$(2.4) \quad \|\Theta_t\|_1 = \|\Theta_t\| = 1, \text{ for } t > 0$$

$$(2.5) \quad \Theta_a * \Theta_b = \Theta_{a+b}, \text{ provided } 1/a + 1/b > 0;$$

$$(2.6) \quad \Theta_a \Theta_b = \frac{\Theta_{ab/(a+b)}}{2\sqrt{\pi}|a+b|^{1/2}}, \text{ provided } a \neq 0, b \neq 0, a+b \neq 0.$$

3. INITIAL DATA IN THE ALEXIEWICZ SPACE

When $f \in \mathcal{A}_c$ the convolution $u(x, t) = f * \Theta_t(x)$ provides a smooth solution of (1.2). The initial conditions are taken on in the Alexiewicz norm (1.4). The estimates $\|f * \Theta_t\|_{\infty} \leq \|f\|/(2\sqrt{\pi t})$ and $\|f * \Theta_t\| \leq \|f\|$ are shown to be sharp.

Widder [29] has written solutions of the heat equation as Stieltjes integrals $f * \Theta_t(x) = \int_{-\infty}^{\infty} \Theta_t(x - \xi) dF(\xi)$ where $F \in \mathcal{BV}$ is the primitive of f . Positive solutions of the heat equation are necessarily given by such an integral with F increasing [29, VIII.3]. Since \mathcal{B}_c contains primitives that are not of bounded variation, solutions considered in Theorem 3.1 below need not be the difference of two positive functions. When F is not absolutely continuous, solutions can be singular at each $x \in \mathbb{R}$ as $t \rightarrow 0+$ [29, XIV.8]. This corresponds to distributions f which have no pointwise values.

Theorem 3.1. *Let $f \in \mathcal{A}_c$. Let the primitive of f be $F \in \mathcal{B}_c$.*

- (a) *The integrals $f * \Theta_t(x) = \Theta_t * f(x) = F * \Theta'_t(x) = [F * \Theta_t]'(x)$ exist for each $x \in \mathbb{R}$ and $t > 0$.*
- (b) *$f * \Theta_t(x)$ is C^∞ for $(x, t) \in \mathbb{R} \times (0, \infty)$.*
- (c) *The estimate $\|f * \Theta_t\|_\infty \leq \|f\|/(2\sqrt{\pi t})$ is sharp in the sense that the coefficient of $\|f\|$ cannot be reduced.*
- (d) *Define the linear operator $\Phi_t : \mathcal{A}_c \rightarrow C(\overline{\mathbb{R}})$ by $\Phi_t(f) = f * \Theta_t$. Then $\|\Phi_t\| = 1/(2\sqrt{\pi t})$.*
- (e) *$\lim_{|x| \rightarrow \infty} f * \Theta_t(x) = 0$.*
- (f) *For each $t > 0$, $f * \Theta_t \in \mathcal{A}_c$ and the inequality $\|f * \Theta_t\| \leq \|f\|$ is sharp in the sense that the coefficient of $\|f\|$ cannot be reduced. Define the linear operator $\Psi_t : \mathcal{A}_c \rightarrow \mathcal{A}_c$ by $\Psi_t(f) = f * \Theta_t$. Then $\|\Psi_t\| = 1$.*
- (g) *Let $u(x, t) = f * \Theta_t(x)$ then u is a solution of (1.1)-(1.2) and (1.4).*
- (h) *For each $t > 0$ we have $\int_{-\infty}^{\infty} f * \Theta_t = \int_{-\infty}^{\infty} f$.*
- (i) *$f * \Theta_t$ need not be in any of the L^p spaces ($1 \leq p < \infty$).*
- (j) *For each $t > 0$, $f * \Theta_t(x)$ is real analytic as a function of $x \in \mathbb{R}$. For each $x \in \mathbb{R}$, $f * \Theta_t(x)$ is real analytic as a function of $t > 0$.*

Proof. (a) The multipliers for \mathcal{A}_c are the functions of bounded variation. See [22]. For each $x \in \mathbb{R}$ and $t > 0$ we have $V_{\xi \in \mathbb{R}} \Theta_t(x - \xi) = 1/\sqrt{\pi t}$ so $f * \Theta_t(x)$ exists for each $x \in \mathbb{R}$ and $t > 0$. The convolution is commutative since we can change variables, $\xi \mapsto x - \xi$ [22, Theorem 11]. Integration by parts (2.2) establishes the other equalities, except the last which follows by Taylor's theorem.

(b) By [23, Theorem 4.1] we can differentiate under the integral sign and this shows $f * \Theta$ is C^∞ in $\mathbb{R} \times (0, \infty)$.

(c) The heat kernel $\xi \mapsto \Theta_t(x - \xi)$ is monotonic on the intervals $(-\infty, x]$ and $[x, \infty)$. By the second mean value theorem for integrals [22, Theorem 26] there are $x_1 \leq x \leq x_2$ such that

$$\begin{aligned} f * \Theta_t(x) &= \Theta_t(\infty) \int_{-\infty}^{x_1} f + \Theta_t(0) \int_{x_1}^x f + \Theta_t(0) \int_x^{x_2} f + \Theta_t(-\infty) \int_{x_2}^{\infty} f \\ &= \frac{1}{2\sqrt{\pi t}} \int_{x_1}^{x_2} f. \end{aligned}$$

It now follows that $\|f * \Theta_t\|_\infty \leq \|f\|/(2\sqrt{\pi t})$. Let $s > 0$ and $f = \Theta_s$. Then $\|f\| = 1$ and $\|\Theta_s * \Theta_t\|_\infty = \|\Theta_{s+t}\|_\infty = 1/[2\sqrt{\pi(s+t)}]$. Letting $s \rightarrow 0$ shows the estimate is sharp.

(d) The operator norm is given by $\|\Phi_t\| = \sup_{\|f\|=1} \|f * \Theta_t\|_\infty \leq 1/(2\sqrt{\pi t})$, by (c). The example in (c) shows $\|\Phi_t\| = 1/(2\sqrt{\pi t})$.

(e) By [23, Theorem 2.1], $\lim_{|x| \rightarrow \infty} f * \Theta_t(x) = \Theta_t(\infty) \int_{-\infty}^{\infty} f = 0$.

(f) The inequality $\|f * g\| \leq \|f\| \|g\|_1$ is proven for $f \in \mathcal{A}_c$ and $g \in L^1$ in [23, Theorem 3.4(a)]. This gives $\|f * \Theta_t\| \leq \|f\| \|\Theta_t\|_1 = \|f\|$. And, $\|\Psi_t\| = \sup_{\|f\|=1} \|f * \Theta_t\| \leq 1$. We get equality by taking $f = \Theta_s$ for $s > 0$. Then $\|\Psi\| \geq \|\Theta_s * \Theta_t\| = \|\Theta_{s+t}\|_1 = 1$.

(g) By (b) we can differentiate under the integral sign. Since the heat kernel is a solution of the heat equation in $\mathbb{R} \times (0, \infty)$ so is $f * \Theta$. Continuity in the Alexiewicz norm gives (1.4). See [23, Theorem 3.4(e)].

(h) Let $-\infty < \alpha < \beta < \infty$. Using the Fubini theorem in the Appendix to [23] and integrating by parts, we have

$$\begin{aligned} \int_{\alpha}^{\beta} f * \Theta_t(x) dx &= \int_{-\infty}^{\infty} f(\xi) \int_{\alpha}^{\beta} \Theta_t(x - \xi) dx d\xi \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \int_{\alpha - \xi}^{\beta - \xi} e^{-x^2/(4t)} dx d\xi \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} F(\xi) \left[e^{-(\beta - \xi)^2/(4t)} - e^{-(\alpha - \xi)^2/(4t)} \right] d\xi \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} [F(\beta - \xi) - F(\alpha - \xi)] e^{-\xi^2/(4t)} d\xi. \end{aligned}$$

Dominated convergence now allows us to take the limits $\alpha \rightarrow -\infty$ and $\beta \rightarrow \infty$ under the integral to get $\int_{-\infty}^{\infty} f * \Theta_t = \int_{-\infty}^{\infty} f$. Notice that there are no improper integrals so the α and β limits give existence of the continuous primitive integral. See Hake's theorem [22, Theorem 25].

(i) Let $s > 0$, $a_n = 1/\log(n+1)$ and $b_n = 2n^2\sqrt{s+t}$. Define $f(x) = \sum (-1)^n a_n \Theta_s(x - b_n)$. If we have $|x| \leq A$ for a real number A then $2\sqrt{\pi s} a_n \Theta_s(x - b_n) \leq \exp(-(b_n - A)^2/(4s))$ for large enough n . The Weierstrass M -test then shows the series defining f converges uniformly on compact sets and $f \in C(\mathbb{R})$. Let $F(x) = \int_{-\infty}^x f$. To prove $f \in \mathcal{A}_c$ show $F \in \mathcal{B}_c$. Clearly, $F(-\infty) = \lim_{x \rightarrow -\infty} F(x) = 0$. And, $F(\infty) = \sum_{n=1}^{\infty} (-1)^n a_n$. Since f is continuous, we just need to show $\lim_{x \rightarrow \infty} F(x) = F(\infty)$. Note that

$$\begin{aligned} |F(x) - F(\infty)| &= \left| \sum_{n=1}^{\infty} (-1)^n a_n \int_x^{\infty} \Theta_s(y - b_n) dy \right| \\ &= \frac{1}{2} \left| \sum_{n=1}^{\infty} (-1)^n a_n \operatorname{erfc} \left(\frac{x - b_n}{2\sqrt{s}} \right) \right|, \end{aligned}$$

where the complementary error function is $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2} dy$. Suppose $b_m < x \leq b_{m+1}$ for some $m \geq 2$. If $n \leq m - 1$ then $x - b_n > b_m - b_{m-1} =$

$2\sqrt{s+t}(2m-1) \rightarrow \infty$ as $m \rightarrow \infty$. And,

$$(3.1) \quad \left| \sum_{n=1}^{m-1} (-1)^n a_n \operatorname{erfc} \left(\frac{x - b_n}{2\sqrt{s}} \right) \right| \leq \sum_{n=1}^{m-1} a_n \operatorname{erfc} \left(\frac{\sqrt{s+t}(2m-1)}{\sqrt{s}} \right).$$

The complementary error function has asymptotic behaviour [11, 8.254] $\operatorname{erfc}(x) \sim \exp(-x^2)/(\sqrt{\pi}x)$ as $x \rightarrow \infty$. Since $|\sum_1^{m-1} a_n| \leq m/\log(2)$ we see that the term in (3.1) has limit 0 as $m \rightarrow \infty$. And, summation by parts and telescoping give

$$\begin{aligned} & \left| \sum_{n=m}^{\infty} (-1)^n a_n \operatorname{erfc} \left(\frac{x - b_n}{2\sqrt{s}} \right) \right| \\ &= \left| \sum_{n=m}^{\infty} \left(\sum_{k=m}^n (-1)^k a_k \right) \left[\operatorname{erfc} \left(\frac{x - b_{n+1}}{2\sqrt{s}} \right) - \operatorname{erfc} \left(\frac{x - b_n}{2\sqrt{s}} \right) \right] \right| \\ &\leq 2 \sup_{N \geq m} \left| \sum_{k=m}^N (-1)^k a_k \right|, \end{aligned}$$

since $0 \leq \operatorname{erfc}(x) \leq 2$. This term has limit 0 as $m \rightarrow \infty$. Hence, $f \in \mathcal{A}_c$.

The Fubini–Tonelli theorem shows we can interchange series and integral to get $f * \Theta_t(x) = \sum (-1)^n a_n \Theta_{s+t}(x - b_n)$. Let $m > 1$ and suppose $|x - b_m| \leq 2\sqrt{s+t}$ then $\Theta_{s+t}(x - b_m) \geq 1/[2e\sqrt{\pi(s+t)}]$. Note that for each $n \geq 1$ we have $b_{n+1} - b_n = 2\sqrt{s+t}(2n+1) > 2\sqrt{s+t}$. The sequence $\Theta_{s+t}(x - b_n)$ is increasing for $b_n \leq x$. Since $x \geq b_m - 2\sqrt{s+t}$, this sequence is increasing for $n^2 \leq m^2 - 1$ and hence for $n < m$. It is decreasing for $n > m$. The series alternates and $a_n \Theta_{s+t}(x - b_n)$ is decreasing, so

$$\begin{aligned} \left| \sum_{n=m+1}^{\infty} (-1)^n a_n \Theta_{s+t}(x - b_n) \right| &\leq a_{m+1} \Theta_{s+t}(b_{m+1} - (b_m + 2\sqrt{s+t})) \\ &\leq \frac{a_m}{2e^{4m^2} \sqrt{\pi(s+t)}}. \end{aligned}$$

Let $\|a\| = \sup_{N \geq 1} |\sum_{n=1}^N (-1)^n a_n|$. Using summation by parts and telescoping, we get

$$\begin{aligned} \left| \sum_{n=1}^{m-1} (-1)^n a_n \Theta_{s+t}(x - b_n) \right| &\leq \left| \left(\sum_{n=1}^{m-1} (-1)^n a_n \right) \Theta_{s+t}(x - b_{m-1}) \right| \\ &\quad + \left| \sum_{n=1}^{m-2} \left(\sum_{k=1}^n (-1)^k a_k \right) [\Theta_{s+t}(x - b_{n+1}) - \Theta_{s+t}(x - b_n)] \right| \\ &\leq 2\|a\| \Theta_{s+t}(b_m - 2\sqrt{s+t} - b_{m-1}) \\ &= \frac{2\|a\| \exp(-4(m-1)^2)}{2\sqrt{\pi(s+t)}} \leq \frac{a_m}{2e^2 \sqrt{\pi(s+t)}} \end{aligned}$$

provided the inequality

$$(3.2) \quad 4(m-1)^2 \geq \log [2\|a\|e^2 \log(m+1)]$$

is satisfied. Clearly we can take $M \geq 2$ large enough so that (3.2) holds for each $m \geq M$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} |f * \Theta_t(x)|^p dx &\geq \sum_{m=M}^{\infty} \int_{|x-b_m| < 2\sqrt{s+t}} |f * \Theta_t(x)|^p dx \\ &\geq \sum_{m=M}^{\infty} \int_{|x-b_m| < 2\sqrt{s+t}} \left| \frac{a_m}{2e\sqrt{\pi(s+t)}} - \frac{a_m}{2e^{4m^2}\sqrt{\pi(s+t)}} - \frac{a_m}{2e^2\sqrt{\pi(s+t)}} \right|^p dx \\ &> 4\sqrt{s+t} \left[\frac{e-2}{2e^2\sqrt{\pi(s+t)}} \right]^p \sum_{m=M}^{\infty} a_m^p = \infty. \end{aligned}$$

(j) Since F is bounded it satisfies the growth condition $|F(x)| \leq C_1 \exp(C_2|x|^{1+\alpha})$ ($C_1 > 0$, $C_2 \in \mathbb{R}$, $0 \leq \alpha < 1$) which ensures $F * \Theta_t(x)$ is real analytic, separately in $x \in \mathbb{R}$ and $t > 0$ [5, Theorem 10.2.1], [5, Theorem 10.3.1]. But then $(F * \Theta_t)' = f * \Theta_t$ is also analytic. \square

Remark 3.2 (Theorem 3.1). Part (a) shows that $f * \Theta_t$ can be written as an improper Riemann integral.

Many results continue to hold in the space $\mathcal{A}_{buc} := \{f = F' \mid F \in \mathcal{B}_{buc}\}$ where \mathcal{B}_{buc} are the functions in $C(\mathbb{R})$ that are bounded and uniformly continuous. The multipliers for \mathcal{A}_{buc} are the functions of bounded variation that have limit 0 at $\pm\infty$, which includes the heat kernel. The space \mathcal{A}_{buc} is a subspace of the weighted spaces studied in Section 5 below. See Example 3.6(d).

When $f \in L^p$ for some $1 \leq p \leq \infty$ the uniform estimate from the Hölder inequality is

$$\|f * \Theta_t\|_{\infty} \leq c_p \|f\|_p t^{-1/(2p)}, \quad c_p = \begin{cases} 1/(2\sqrt{\pi}), & p = 1 \\ (4\pi)^{-1/(2p)} [(p-1)/p]^{(p-1)/(2p)}, & 1 < p < \infty \\ 1, & p = \infty. \end{cases}$$

For example, [13]. The condition for equality in the Hölder inequality [17, p. 46] shows these estimates are sharp. When $1 < p < \infty$ we can take $f = \Theta_t^{q/p}$. When $p = \infty$ we can take $f = 1$. When $p = 1$ the condition is $\Theta_t(x) = d \operatorname{sgn}[f(x)]$ for some $d \in \mathbb{R}$. This cannot be satisfied with any $f \in L^1$. Instead, we take for a delta sequence. Note that when $f \in L^1$ we get the same estimate as for $f \in \mathcal{A}_c$ (Theorem 3.1(c)), with the Alexiewicz norm replaced with the L^1 norm.

Part (h) of Theorem 3.1 is given for $f \in L^1$ in [7, p. 220].

Note that since $f * \Theta_t$ is continuous the integral $\int_{-\infty}^{\infty} f * \Theta_t$ exists as a Henstock–Kurzweil integral and as an improper Riemann integral but from (i) $\int_{-\infty}^{\infty} |f * \Theta_t(x)| dx$ can diverge.

Theorem 10.2.1 in [5] shows $f * \Theta_t(x)$ is separately analytic in $x \in \mathbb{C}$ and in a domain with bounded imaginary part of t .

We have continuity with respect to initial conditions.

Corollary 3.3. *Suppose $f, g \in \mathcal{A}_c$.*

(a) *Then $\|f * \Theta_t - g * \Theta_t\| \leq \|f - g\|$ for each $t > 0$.*

- (b) Let $\epsilon > 0$. Suppose $u, v : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ such that $\|u(\cdot, t) - f\| \rightarrow 0$ and $\|v(\cdot, t) - g\| \rightarrow 0$ as $t \rightarrow 0^+$. If $\|f - g\| < \epsilon$ then for small enough t we have $\|u(\cdot, t) - v(\cdot, t)\| < 2\epsilon$.

Proof. Part (a) follows from (f) and part (b) is a consequence of the triangle inequality. \square

Note that in (b) u and v need not be solutions of the heat equation and need not be convolutions of the heat kernel with f or g . See [13] for corresponding results in L^p spaces.

Remark 3.4 (The regulated primitive integral and L^p primitive integral). A regulated integral is obtained by taking left continuous regulated functions as primitives. A function is regulated on \mathbb{R} if it has left and right limits at each point of \mathbb{R} and it is left continuous on $\overline{\mathbb{R}}$ if it equals its left limit at each point of $(-\infty, \infty]$ and it has a limit at $-\infty$. If F is such a function then the regulated primitive integral of F' is $\int_{-\infty}^{\infty} F' = F(\infty)$. There are four types of integral on finite intervals: $\int_{[a,b]} f = F(b+) - F(a-) = F(b+) - F(a)$, $\int_{(a,b]} f = F(b+) - F(a+)$, $\int_{[a,b)} f = F(b-) - F(a-) = F(b) - F(a)$, $\int_{(a,b)} f = F(b-) - F(a+) = F(b) - F(a+)$. The space of integrable distributions is a Banach space under the Alexiewicz norm. This integral contains the continuous primitive integral. As well, all finite signed Borel measures are integrable. For example, the Dirac measure is the distributional derivative of the Heaviside step function $H = \chi_{(0,\infty]}$. The regulated primitive integral is discussed in [24]. While many of the properties proven in Theorem 3.1 continue to hold, a key difference between the two integrals is that, for the regulated primitive integral, translation is not continuous in the Alexiewicz norm. A consequence of this is that $f * \Theta_t$ need not converge to f as $t \rightarrow 0^+$. For example, $\delta * \Theta_t(x) = \Theta_t(x)$ and $\|\delta * \Theta_t - \delta\| \geq |\int_{(0,\infty)} (\Theta_t - \delta)| = 1/2 \not\rightarrow 0$ as $t \rightarrow 0^+$. For this reason, we do not consider the regulated primitive integral in this paper. In Section 4 we define higher order Alexiewicz spaces which contain some linear combinations of translated Dirac distributions. In all these spaces we get convolutions with the initial data converging to the initial data in the appropriate norm.

We do have continuity in the L^p norms for $1 \leq p < \infty$. Distributions that are the distributional derivative of an L^p function are considered in [26]. When these are convolved with the heat kernel they give solutions that take on initial conditions in spaces of distributions that are isometrically isomorphic to L^p .

When f is locally a continuous function, $f * \Theta_t(x)$ converges pointwise to $f(x)$ as $t \rightarrow 0^+$.

Theorem 3.5. Suppose $f \in \mathcal{A}_c$ with primitive $F \in \mathcal{B}_c$ such that $F|_I \in C^1(I)$ for some open interval $I \subset \mathbb{R}$. Define $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ by

$$u(x, t) = \begin{cases} f * \Theta_t(x), & (x, t) \in \mathbb{R} \times (0, \infty) \\ f(x), & (x, t) \in I \times \{0\} \\ 0, & (x, t) \in (\mathbb{R} \setminus I) \times \{0\}. \end{cases}$$

Then u is continuous on $[\mathbb{R} \times (0, \infty)] \cup [I \times \{0\}]$.

Proof. Continuity of u on $\mathbb{R} \times (0, \infty)$ is proved in Theorem 3.1(b). Let $x_0 \in I$. Given $\epsilon > 0$ there exists $\delta > 0$ such that $(x_0 - 2\delta, x_0 + 2\delta) \subset I$ and if $|x - x_0| < 2\delta$ then $|f(x) - f(x_0)| < \epsilon$. For $(x, t) \in (x_0 - \delta, x_0 + \delta) \times (0, \infty)$ we have

$$\begin{aligned} |f * \Theta_t(x) - f(x_0)| &= \left| \int_{-\infty}^{\infty} f(\xi) \Theta_t(x - \xi) d\xi - f(x_0) \int_{-\infty}^{\infty} \Theta_t(x - \xi) d\xi \right| \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \left| \int_{x_0 - 2\delta}^{x_0 + 2\delta} [f(\xi) - f(x_0)] \Theta_t(x - \xi) d\xi \right| \leq \epsilon \int_{-\infty}^{\infty} \Theta_t(x - \xi) d\xi = \epsilon, \\ I_2 &= \left| \int_{|\xi - x_0| > 2\delta} f(\xi) \Theta_t(x - \xi) d\xi \right| \\ &\leq \|f\| \{V_{\xi \in \mathbb{R}}[\Theta_t(x - \xi) \chi_{(x_0 + 2\delta, \infty)}(\xi)] + V_{\xi \in \mathbb{R}}[\Theta_t(x - \xi) \chi_{(-\infty, x_0 - 2\delta)}(\xi)]\} \\ &\leq 2\|f\| [\Theta_t(\delta) + \Theta_t(-\delta)] \\ I_3 &= |f(x_0)| \int_{|\xi - x_0| > 2\delta} \Theta_t(x - \xi) d\xi \\ &\leq \frac{|f(x_0)|}{\sqrt{\pi}} \left(\int_{-\infty}^{-\delta/(2\sqrt{t})} e^{-s^2} ds + \int_{\delta/(2\sqrt{t})}^{\infty} e^{-s^2} ds \right). \end{aligned}$$

The estimate for I_2 comes from the Hölder inequality (2.3). Letting $t \rightarrow 0^+$ shows $I_2 \rightarrow 0$ and $I_3 \rightarrow 0$. \square

Examples 3.6. (a) L^1 , the spaces of Henstock–Kurzweil integrable functions and wide Denjoy integrable functions are all (non-closed) subspaces of \mathcal{A}_c . In each case, $f * \Theta_t(x)$ agrees with its traditional value and its value in \mathcal{A}_c .

(b) Let $V \in \mathcal{B}_c$ be a singular function, i.e., the pointwise derivative $V'(x) = 0$ for almost all x . Then $V'(x) \in L^1$ and the Lebesgue integral gives $V' * \Theta_t(x) = 0$ for each $x \in \mathbb{R}$. This gives the zero solution of the heat equation with zero initial condition taken on in the L^1 norm. As an integral in \mathcal{A}_c we get $V' * \Theta_t(x) = V * \Theta'_t(x) = \int_{-\infty}^{\infty} V(\xi) \Theta'_t(x - \xi) d\xi$. This last exists as an improper Riemann integral and gives a nontrivial solution of the heat equation, initial values being taken on in the Alexiewicz norm.

(c) Suppose $f \in \mathcal{A}_c$ has compact support in the interval $[A, B]$. Let $x > \max(B, 0)$. Then $f * \Theta_t(x) = 1/(2\sqrt{\pi t}) \int_A^B f(\xi) \exp(-(\xi - x)^2/(4t)) d\xi$. By Hölder's inequality (2.3),

$$|f * \Theta_t(x)| \leq \frac{\|f\|}{2\sqrt{\pi t}} V(\Theta_t \chi_{[A, B]}) = \frac{\|f\|}{2\sqrt{\pi t}} \left[e^{-(x-A)^2/(4t)} + e^{-(x-B)^2/(4t)} \right].$$

Hence, for fixed $t > 0$, $f * \Theta_t(x) = O(\exp(-(x - B)^2/(4t)))$ as $x \rightarrow \infty$.

(d) Suppose $f = F'$ where F is bounded and uniformly continuous on \mathbb{R} (but not necessarily on $\overline{\mathbb{R}}$). We can define $\|f\| = \sup_I |\int_I f|$, provided the supremum is taken over bounded intervals $I \subset \mathbb{R}$. The estimates in (c) and (f) of Theorem 3.1 then continue to hold, even though f need not be in \mathcal{A}_c . For example, let $f(x) = \sin(sx)$ for $s > 0$. A routine calculation (also see Example 5.4(c) below)

gives the separated solution $u(x, t) = f * \Theta_t(x) = \sin(sx) \exp(-s^2 t)$. Note that $f \notin \mathcal{A}_c$ but $\|f\| = (1/s) \int_0^\pi \sin(x) dx = 2/s$. And, $\|u_t\|_\infty = \exp(-s^2 t)$. A calculation gives $2\sqrt{\pi t} \|u_t\|_\infty / \|f\| = s\sqrt{\pi t} \exp(-s^2 t) \leq \sqrt{\pi/(2e)} < 1$, in accordance with (c) of Theorem 3.1. And, $\|u_t\| = \exp(-s^2 t) \|f\| \leq \|f\|$, in accordance with (f) of Theorem 3.1. As well, $\|u_t - f\| = (1 - \exp(-s^2 t)) \|f\| \rightarrow 0$ as $t \rightarrow 0^+$.

(e) Let $f(x) = \exp(ix^2/(4s))$ for $s > 0$. Then f is bounded and continuous but not in the L^p spaces ($1 \leq p < \infty$). And, f is improper Riemann integrable so $f \in \mathcal{A}_c$. Completing the square gives

$$\begin{aligned} f * \Theta_t(x) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(\xi-x)^2/(4t)} e^{i\xi^2/(4s)} d\xi \\ &= \frac{e^{-\frac{x^2}{4t}} e^{\frac{sx^2(s+it)}{4t(s^2+t^2)}}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-it)}{4st} \left[\xi - \frac{s(s+it)x}{s^2+t^2} \right]^2} d\xi. \end{aligned}$$

Due to Cauchy's theorem, the contour can be shifted in the complex plane from ξ real to $\mathcal{I}m(\xi) = stx/(s^2 + t^2)$. The integral $\int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = \sqrt{\pi/a}$ for $\mathcal{R}e(a) > 0$ then gives

$$(3.3) \quad f * \Theta_t(x) = \frac{e^{-\frac{tx^2}{4(s^2+t^2)}} e^{\frac{isx^2}{4(s^2+t^2)}} \sqrt{s} \sqrt{s+it}}{\sqrt{s^2+t^2}}.$$

It is readily seen that $\lim_{t \rightarrow 0^+} f * \Theta_t(x) = f(x)$ at each point $x \in \mathbb{R}$ (cf. Theorem 3.5). Differentiating f and $f * \Theta_t$ with respect to s gives an example for which the initial data is not in \mathcal{A}_c . See Example 5.4(e) below.

Note that for fixed $s, t > 0$ there is a positive constant c such that $f * \Theta_t(x) = O(e^{-cx^2})$ as $|x| \rightarrow \infty$. The next example shows that $f * \Theta_t(x)$ can tend to 0 more slowly (cf. Theorem 3.1(e)).

(f) Suppose a decay rate $\omega : (0, \infty) \rightarrow (0, \infty)$ is given and $\lim_{x \rightarrow \infty} \omega(x) = 0$. We will show there is $f \in L^p$ for each $1 \leq p \leq \infty$ such that $f * \Theta_t(x) \neq o(\omega(x))$ as $x \rightarrow \infty$. We can assume ω is decreasing; otherwise replace ω with $\tilde{\omega}(x) = \sup_{t > x} \omega(t)$. Let $f(x) = \sum_{n=1}^{\infty} n^{-2} \Theta_s(x - b_n)$ where $s > 0$ and $\{b_n\}$ is an increasing sequence with limit ∞ . Then $f \in L^p$. And, $f * \Theta_t(b_m) \geq m^{-2} \Theta_{s+t}(0) = [2m^2 \sqrt{\pi(s+t)}]^{-1}$. Let $b_n = \omega^{-1}(n^{-2})$. Then $b_n \uparrow \infty$. And,

$$\frac{f * \Theta_t(b_m)}{\omega(b_m)} \geq \frac{1}{2\sqrt{\pi(s+t)}} \not\rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

We say $f \in \mathcal{A}_c$ has an absolutely convergent integral if f has a primitive $F \in \mathcal{BV} \cap \mathcal{B}_c$.

Proposition 3.7. (a) Let $f \in \mathcal{A}_c$ with primitive $F \in \mathcal{BV} \cap \mathcal{B}_c$. Then $V(f * \Theta_t) \leq VFV\Theta_t = VF/\sqrt{\pi t}$.

(b) Let $f \in \mathcal{A}_c$ with compact support in $[A, B]$. Then

$$V(f * \Theta_t) \leq \frac{\|f\|}{\sqrt{\pi}} \left[\frac{1}{\sqrt{t}} + \frac{(B-A)\sqrt{2}}{\sqrt{e}t} \right].$$

- (c) If $f \in L^1$ then $V(f * \Theta_t) \leq \|f\|_1 / \sqrt{\pi t}$.
 (d) Let $f \in \mathcal{A}_c$ and let $u(x, t) = u_t(x) = f * \Theta_t(x)$. For $t > 0$ the pointwise derivatives have estimates $\|u'_t\| \leq \|f\| V\Theta_t = \|f\| / \sqrt{\pi t}$, $\|\partial u(\cdot, t) / \partial t\| = \|u''_t\| \leq \|f\| V\Theta'_t = \|f\| \sqrt{2} / (\sqrt{\pi e} t)$, $\|u'_t\|_\infty \leq \|f\| V\Theta'_t = \|f\| \sqrt{2} / (\sqrt{\pi e} t)$, $\|\partial u(\cdot, t) / \partial t\|_\infty = \|u''_t\|_\infty \leq \|f\| V\Theta''_t = \|f\| (1 + 4e^{-3/2}) / (2\sqrt{\pi} t^{3/2})$.

Proof. (a) Let (x_i, y_i) be a sequence of disjoint intervals. Integrating by parts and using the Fubini–Tonelli theorem

$$\begin{aligned} \sum_i |f * \Theta_t(x_i) - f * \Theta_t(y_i)| &= \sum_i \left| \int_{-\infty}^{\infty} \Theta'_t(\xi) [F(x_i - \xi) - F(y_i - \xi)] d\xi \right| \\ &\leq \int_{-\infty}^{\infty} |\Theta'_t(\xi)| \sum_i |F(x_i - \xi) - F(y_i - \xi)| d\xi \\ &\leq V\Theta_t VF. \end{aligned}$$

Taking the supremum over all such intervals gives the first result. (b) Let the primitive of f be F . Then $F = 0$ on $(-\infty, A]$ and $F = F(B)$ on $[B, \infty)$. Integrate by parts and use the Fubini–Tonelli theorem in [23] to get

$$\begin{aligned} V(f * \Theta_t) &= \int_{-\infty}^{\infty} |(f * \Theta_t)'(x)| dx \\ &= \int_{-\infty}^{\infty} \left| F(B)\Theta'_t(x - B) + \int_A^B F(\xi)\Theta''_t(x - \xi) d\xi \right| dx \\ &\leq |F(B)| / (\sqrt{\pi t}) + (B - A) \|F\|_\infty V\Theta'_t \\ &\leq \frac{\|f\|}{\sqrt{\pi}} \left[\frac{1}{\sqrt{t}} + \frac{(B - A)\sqrt{2}}{\sqrt{e} t} \right]. \end{aligned}$$

(c) This follows since if $f \in L^1$ then it has a primitive that is absolutely continuous and of bounded variation. (d) The heat kernel is smooth so we can differentiate pointwise [23, Corollary 4.5] to get $\partial u(x, t) / \partial x = u'_t(x) = f * \Theta'_t(x)$, $\partial^2 u(x, t) / \partial x^2 = u''_t(x) = \partial u(x, t) / \partial t = f * \Theta''_t(x)$. A calculation shows $V\Theta'_t = 4\Theta'_t(-\sqrt{2t}) = \sqrt{2} / (\sqrt{\pi e} t)$ and $V\Theta''_t = 4\Theta''_t(\sqrt{6t}) - 2\Theta''_t(0) = (1 + 4e^{-3/2}) / (2\sqrt{\pi} t^{3/2})$. The result then follows from the estimates $\|f * g\| \leq \|f\| \|g\|_1$ ([23, Theorem 3.4]) and the Hölder inequality (2.3). \square

4. HIGHER ORDER ALEXIEWICZ SPACES

For each $n \in \mathbb{N}$ the set of distributions that are the n th derivative of a function in \mathcal{B}_c is a Banach space isometrically isomorphic to \mathcal{B}_c . A distributional integral is defined, the multipliers being functions that are n -fold iterated integrals of functions of bounded variation. Properties analogous to Theorem 3.1 hold for these higher order distributions. First we briefly introduce the higher order distributional integrals. More details can be found in [25]. Our presentation is simplified since we take primitives to be continuous rather than regulated functions, c.f. Remark 3.4.

If $F, G \in \mathcal{B}_c$ and satisfy the distributional differential equation $F^{(n)} = G^{(n)}$ then F and G differ by a polynomial of degree at most $n - 1$. But the only polynomial in \mathcal{B}_c is the zero function so $F = G$. This shows that according to the following definition the primitive of a distribution in \mathcal{A}_c^n is unique.

Definition 4.1. For each $n \in \mathbb{N}$ let $\mathcal{A}_c^n = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F^{(n)} \text{ for some } F \in \mathcal{B}_c\}$, with $\mathcal{A}_c^1 := \mathcal{A}_c$. If $f \in \mathcal{A}_c^n$ define $\|f\|^{(n)} = \|F'\| = \sup_{x < y} |F(x) - F(y)|$ where $F \in \mathcal{B}_c$ is the unique primitive of f .

It is clear that $\|\cdot\|^{(n)}$ is a norm on \mathcal{A}_c^n and this makes \mathcal{A}_c^n into a Banach space that is isometrically isomorphic to \mathcal{B}_c .

Definition 4.2. Let \mathcal{NBV} be the functions in \mathcal{BV} that are right continuous on $\overline{\mathbb{R}}$, i.e., $g \in \mathcal{BV}$ and $\lim_{y \rightarrow x^+} g(y) = g(x)$ for each $x \in (-\infty, \infty)$. Define $\mathcal{IBV}^0 = \mathcal{NBV}$. Suppose \mathcal{IBV}^{n-1} is known for $n \in \mathbb{N}$. Define $\mathcal{IBV}^n = \{h : \mathbb{R} \rightarrow \mathbb{R} \mid h(x) = \int_0^x q(t) dt \text{ for some } q \in \mathcal{IBV}^{n-1}\}$.

This inductive definition defines the multipliers for integration in \mathcal{A}_c^n . Each function in $h \in \mathcal{IBV}^n$ is an n -fold iterated integral of a unique function $g \in \mathcal{NBV}$. We write this as $h = I^{n-1}[g]$. See [25], Proposition 2.5 and the remark following. Note that $h^{(k)}(0) = 0$ for $0 \leq k \leq n - 2$. Now we can define the integral of the product of a distribution in \mathcal{A}_c^n and a function in \mathcal{IBV}^{n-1} .

Definition 4.3. Let $n \in \mathbb{N}$. For $f \in \mathcal{A}_c^n$ let F be its primitive in \mathcal{B}_c . For $h \in \mathcal{IBV}^{n-1}$ such that $h = I^{n-1}[g]$ for $g \in \mathcal{NBV}$, define the continuous primitive integral of f with respect to h as

$$(4.1) \quad \int_{-\infty}^{\infty} fh = \int_{-\infty}^{\infty} F^{(n)}h = (-1)^{n-1} \int_{-\infty}^{\infty} F'h^{(n-1)}$$

$$(4.2) \quad = (-1)^{n-1} F(\infty)g(\infty) - (-1)^{n-1} \int_{-\infty}^{\infty} F(x) dg(x).$$

Note that $F' \in \mathcal{A}_c$ and $h^{(n-1)} \in \mathcal{BV}$ so that the last integral in (4.1) is integration in \mathcal{A}_c as in Section 2. It is shown in [25, Proposition 2.10] that if P is a polynomial of degree at most $n - 2$ then $\int_{-\infty}^{\infty} fP = 0$. Hence, the iterated integrals defining h can have any point $a \in \overline{\mathbb{R}}$ as their lower limit and $h^{(k)}$ would vanish at a for $0 \leq k \leq n - 2$ and not necessarily at 0.

The Hölder inequality now takes the following form.

Proposition 4.4. Let $f \in \mathcal{A}_c^n$ with primitive $F \in \mathcal{B}_c$. Let $h \in \mathcal{IBV}^{n-1}$ such that $h = I^{n-1}[g]$ for $g \in \mathcal{NBV}$. Then

$$\left| \int_{-\infty}^{\infty} fh \right| \leq \left| \int_{-\infty}^{\infty} F' \right| \inf |g| + \|F'\| Vg \leq \|f\|^{(n)} [\inf |h^{(n-1)}| + Vh^{(n-1)}].$$

Every derivative of the heat kernel is a function of bounded variation vanishing at $\pm\infty$. Hence, if $f \in \mathcal{A}_c^n$ then $f * \Theta_t$ exists. This leads to similar results as in Theorem 3.1.

Lemma 4.5. *Let $n \in \mathbb{N}$. Then $\Theta_t^{(n)}(x) = (-2\sqrt{t})^{-n}\Theta_t(x)H_n(x/(2\sqrt{t}))$ and $V\Theta_t^{(n-1)} = \|\Theta_t^{(n)}\|_1 \leq c_n t^{-n/2}$ where*

$$c_n = \frac{1}{2^n \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} |H_n(x)| dx \leq k \sqrt{n!} 2^{(1-n)/2}$$

and H_n is the n th order Hermite polynomial. It is known that $k < 1.087$.

Proof. This follows from the Rodrigues formula for Hermite polynomials [11, 8.950.1] and Cramer's inequality for Hermite polynomials [11, 8.954.2]. \square

Theorem 4.6. *Let $n \geq 2$ and let $f \in \mathcal{A}_c^n$. Let the primitive of f be $F \in \mathcal{B}_c$.*

- (a) *The integrals $f * \Theta_t(x) = \Theta_t * f(x) = F * \Theta_t^{(n)}(x) = [F * \Theta_t]^{(n)}(x)$ exist for each $x \in \mathbb{R}$ and $t > 0$.*
- (b) *$f * \Theta_t(x)$ is C^∞ for $(x, t) \in \mathbb{R} \times (0, \infty)$.*
- (c) *Let c_n be as in Lemma 4.5 then $\|f * \Theta_t\|_\infty \leq c_n \|f\|^{(n)} t^{-n/2}$. The exponent on t cannot be changed.*
- (d) *Define the linear operator $\Phi_t : \mathcal{A}_c^n \rightarrow C(\overline{\mathbb{R}})$ by $\Phi_t(f) = f * \Theta_t$. Then $\|\Phi_t\| \leq c_n t^{-n/2}$.*
- (e) *$\lim_{|x| \rightarrow \infty} f * \Theta_t(x) = 0$.*
- (f) *For each $t > 0$, $f * \Theta_t \in \mathcal{A}_c^n$ and the inequality $\|f * \Theta_t\|^{(n)} \leq \|f\|^{(n)}$ is sharp in the sense that the coefficient of $\|f\|^{(n)}$ cannot be reduced. Define the linear operator $\Psi_t : \mathcal{A}_c^n \rightarrow \mathcal{A}_c^n$ by $\Psi_t(f) = f * \Theta_t$. Then $\|\Psi_t\| = 1$. For each integer $0 \leq k \leq n$, $f * \Theta_t \in \mathcal{A}_c^k$ and $\|f * \Theta_t\|^{(k)} \leq \|F'\| c_{n-k+1} t^{(n-k+1)/2}$.*
- (g) *Let $u(x, t) = f * \Theta_t(x)$ then u is a solution of (1.1)-(1.2) and $\|u_t - f\|^{(n)} \rightarrow 0$ as $t \rightarrow 0^+$.*
- (h) *For each $t > 0$ we have $\int_{-\infty}^{\infty} f * \Theta_t = 0$.*
- (i) *For each $t > 0$, $f * \theta_t(x)$ is real analytic as a function of $x \in \mathbb{R}$. For each $x \in \mathbb{R}$, $f * \Theta_t(x)$ is real analytic as a function of $t > 0$.*

Proof. From (4.1) we get $f * \Theta_t(x) = F' * \Theta_t^{(n-1)}(x)$. This then reduces the convolution of f with Θ_t in \mathcal{A}_c^n to convolution of F' with $\Theta_t^{(n-1)}$ in \mathcal{A}_c . Now using Definition 4.1, the corresponding results of Theorem 3.1 apply. (a) A linear change of variables theorem is proved in [25, Theorem 3.4]. This shows the convolution commutes. (c) We have

$$\|f * \Theta_t\|_\infty = \|F * \Theta_t^{(n)}\|_\infty = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{\infty} F(\xi) \Theta_t^{(n)}(x - \xi) d\xi \right| \leq \|f\|^{(n)} \|\Theta_t^{(n)}\|_1.$$

Now use Lemma 4.5. Let $s > 0$ and take $f(x) = \Theta_s^{(n-1)}(x)$. Then

$$f * \Theta_t(x) = [\Theta_{s+t}]^{(n-1)}(x) = \frac{(-1)^{n-1} \Theta_{s+t}(x) H_{n-1}(x/(2\sqrt{s+t}))}{2^{n-1} (s+t)^{(n-1)/2}}.$$

Let $\alpha_n \in \mathbb{R}$ such that $H_{n-1}(\alpha_n) \neq 0$. Then

$$\|f * \Theta_t\|_\infty \geq |f * \Theta_t(2\alpha_n \sqrt{s+t})| = \frac{\exp(-\alpha_n^2) H_{n-1}(\alpha_n)}{2^n \sqrt{\pi} (s+t)^{n/2}}.$$

Letting $s \rightarrow 0$ shows the exponent on t cannot be changed. (e) Using Lemma 4.5, dominated convergence gives

$$\lim_{x \rightarrow \pm\infty} f * \Theta_t(x) = \lim_{x \rightarrow \pm\infty} \int_{-\infty}^{\infty} F(x - \xi) \Theta_t^{(n)}(\xi) d\xi = F(\mp\infty) \int_{-\infty}^{\infty} \Theta_t^{(n)}(\xi) d\xi = 0.$$

(f) To show the inequality is sharp let $f = \Theta_s^{(n-1)}$. Note that for each integer $k \leq n$ we have $F * \Theta_t^{(n-k)} \in C(\overline{\mathbb{R}})$. And, $f * \Theta_t = (F * \Theta_t^{n-k})^{(k)} \in \mathcal{A}_c^k$. (h) Similar analysis as in the proof of Theorem 3.1 shows

$$\begin{aligned} \int_{\alpha}^{\beta} f * \Theta_t(x) dx &= \int_{-\infty}^{\infty} F'(\xi) \int_{\alpha}^{\beta} \Theta_t^{(n-1)}(x - \xi) dx d\xi \\ &= \int_{-\infty}^{\infty} [F(\beta - \xi) - F(\alpha - \xi)] \Theta_t^{(n-1)}(\xi) d\xi. \end{aligned}$$

Dominated convergence then gives

$$\int_{-\infty}^{\infty} f * \Theta_t(x) dx = F(\infty) \int_{-\infty}^{\infty} \Theta_t^{(n-1)}(\xi) d\xi = 0.$$

□

Remark 4.7 (Theorem 4.6). (c) The sharp constant is not known for $n \geq 2$.

Examples 4.8. (a) Let w be a bounded continuous function such that the pointwise derivative $w'(x)$ exists for no x . Define $F_1(x) = w(x) \exp(-|x|)$ then $F_1 \in \mathcal{B}_c$. For $n \geq 1$, $F_1^{(n)}$ has pointwise values nowhere and yet $F_1^{(n)} * \Theta_t$ is well-defined as an integral in \mathcal{A}_c^n and defines a smooth solution of the heat equation, taking on initial values in the norm $\|\cdot\|^{(n)}$, as in Theorem 4.6. Note that as a Lebesgue integral $F_1^{(n)} * \Theta_t$ does not exist.

(b) Neither the Dirac distribution nor any of its derivatives are in \mathcal{A}_c^n [25, Proposition 3.1]. However, some linear combinations of the Dirac distribution are in \mathcal{A}_c^n . Define $F_2(x) = 0$ for $x \leq 0$, $F_2(x) = x$ for $0 \leq x \leq 1$, $F_2(x) = 1$ for $x \geq 1$. Then $F_2 \in \mathcal{B}_c$. Let $\delta_a = \tau_a \delta$, the Dirac distribution supported at $a \in \mathbb{R}$, with $\delta_0 = \delta$. Then $F_2'(x) = H(x) - H(x - 1)$ where H is the Heaviside step function. For $n \geq 2$ we get $F_2^{(n)} = \delta^{(n-2)} - \delta_1^{(n-2)} \in \mathcal{A}_c^n$. A solution of the heat equation is then given using Lemma 4.5

$$\begin{aligned} u(x, t) &= F_2^{(n)} * \Theta_t(x) = \Theta_t^{(n-2)}(x) - \Theta_t^{(n-2)}(x - 1) \\ &= (-2\sqrt{t})^{-(n-2)} \left[\Theta_t(x) H_{n-2} \left(\frac{x}{2\sqrt{t}} \right) - \Theta_t(x - 1) H_{n-2} \left(\frac{x - 1}{2\sqrt{t}} \right) \right]. \end{aligned}$$

And, $\|u_t - F_2^{(n)}\|^{(n)} \rightarrow 0$ as $t \rightarrow 0^+$.

(c) Fix $\alpha > 0$. Let $p_{\alpha}(x) = x^{\alpha}$ and $q(x) = \exp(-x)$. Define $F_3 = H p_{\alpha} q \in \mathcal{B}_c$. The pointwise derivative satisfies $F_3^{(n)}(x) \sim \alpha(\alpha - 1) \cdots (\alpha - n + 1) x^{\alpha-n}$ as $x \rightarrow 0^+$. Hence, for each $n \in \mathbb{N}$ we have $F_3^{(n)} \in \mathcal{A}_c^n$ and for $n \geq \alpha + 1$ we have $F_3^{(n)} \notin L_{loc}^1$. Although $F_3^{(n)}$ need not be locally integrable in the Lebesgue (or Henstock–Kurzweil) sense, $F_3^{(n)} * \Theta_t$ gives a smooth solution of the heat equation, taking on initial values in the norm $\|\cdot\|^{(n)}$, as in Theorem 4.6.

Proposition 4.9. *Let c_n be as in Lemma 4.5.*

- (a) *Let $f \in \mathcal{A}_c^n$ with primitive $F \in \mathcal{BV} \cap \mathcal{B}_c$. Then $V(f * \Theta_t) \leq VF \|\Theta_t^{(n)}\|_1 = c_n VF t^{-n/2}$.*
- (b) *Let $f \in \mathcal{A}_c^n$ with compact support in $[A, B]$. Then $V(f * \Theta_t) \leq \|f\|^{(n)} [c_n t^{-n/2} + (B - A)c_{n+1} t^{-(n+1)/2}]$.*
- (c) *If $f = F^{(n)}$ where $F \in AC \cap \mathcal{BV}$ then $V(f * \Theta_t) \leq c_n \|F'\|_1 t^{-n/2}$.*
- (d) *Let $f \in \mathcal{A}_c^n$ and let $u(x, t) = u_t(x) = f * \Theta_t(x)$. For $t > 0$ the point-wise derivatives have estimates $\|u_t'\|^{(n)} \leq \|f\|^{(n)} V\Theta_t = \|f\|^{(n)} / \sqrt{\pi t}$, $\|\partial u(\cdot, t) / \partial t\|^{(n)} = \|u_t''\|^{(n)} \leq \|f\|^{(n)} V\Theta_t' = \|f\|^{(n)} \sqrt{2} / (\sqrt{\pi e} t)$, $\|u_t'\|_\infty \leq \|f\|^{(n)} V\Theta_t^{(n)} = c_{n+1} \|f\|^{(n)} t^{-(n+1)/2}$, $\|\partial u(\cdot, t) / \partial t\|_\infty = \|u_t''\|_\infty \leq \|f\|^{(n)} V\Theta_t^{(n+1)} = c_{n+2} \|f\|^{(n)} t^{-(n+2)/2}$.*

Proof. Using Theorem 4.6, the proof is similar to the proof of Proposition 3.7. \square

5. WEIGHTED SPACES

The minimal condition for existence of the convolution $f * \Theta_t(x)$ is that the integral $\int_{-\infty}^{\infty} f(x) e^{-x^2/(4\tau)} dx$ exist for some $\tau > 0$. Then $f * \Theta_t(x)$ exists for $0 < t < \tau$.

Let $G \in \mathcal{B}_c$. Let $\tau > 0$ and define $\omega_\tau(x) = \exp(-x^2/(4\tau))$. Let F_τ be the unique continuous solution of the Volterra integral equation

$$(5.1) \quad G(x) - G(0) = F_\tau(x) \omega_\tau(x) + \frac{1}{2\tau} \int_0^x F_\tau(\xi) \xi \omega_\tau(\xi) d\xi.$$

Necessarily, $F_\tau(0) = 0$. Since the kernel is continuous there is a unique solution for $F_\tau \omega_\tau$ and hence for F_τ . Uniqueness is proven in [30]. If $G_i \in \mathcal{B}_c$ and

$$G_i(x) - G_i(0) = F_\tau(x) \omega_\tau(x) + \frac{1}{2\tau} \int_0^x F_\tau(\xi) \xi \omega_\tau(\xi) d\xi$$

for $i = 1, 2$ then $G_1(x) - G_2(x) = G_1(0) - G_2(0)$ so $G_1 - G_2 \in \mathcal{B}_c$ but is constant so $G_1 = G_2$. Hence, there is a one-to-one correspondence between F_τ and G in (5.1). We then have the following definitions for a family of weighted Alexiewicz spaces.

Definition 5.1. Let $\tau > 0$. Define $\omega_\tau : \mathbb{R} \rightarrow \mathbb{R}$ by $\omega_\tau(x) = e^{-x^2/(4\tau)}$. Let $\mathcal{B}_{c,\tau} = \{F_\tau \in C(\mathbb{R}) \mid F_\tau \text{ is a solution of (5.1) for some } G \in \mathcal{B}_c\}$. For $F_\tau \in \mathcal{B}_{c,\tau}$ define $\|F_\tau\|_{\tau,\infty} = \|G\|'_\infty$ where G is the unique function in (5.1). Define $\mathcal{A}_{c,\tau} = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F'_\tau \text{ for some } F_\tau \in \mathcal{B}_{c,\tau}\}$. For $f \in \mathcal{A}_{c,\tau}$ define $\|f\|_\tau = \|F_\tau\|_{\tau,\infty}$ where F_τ is the unique primitive of f .

For the definition to be meaningful we still need to show distributions in $\mathcal{A}_{c,\tau}$ have a unique primitive in $\mathcal{B}_{c,\tau}$. Suppose $f \in \mathcal{A}_{c,\tau}$ and $f = F'_1 = F'_2$ for $F_1, F_2 \in \mathcal{B}_{c,\tau}$. As before, $F_1 - F_2 = c = \text{constant}$. There are unique $G_i \in \mathcal{B}_c$ such that

$$G_i(x) - G_i(0) = F_i(x) \omega_\tau(x) + \frac{1}{2\tau} \int_0^x F_i(\xi) \xi \omega_\tau(\xi) d\xi$$

for $i = 1, 2$. Integrating shows $G_1(x) - G_2(x) - G_1(0) + G_2(0) = c \omega_\tau(0)$. Putting $x = 0$ shows $c = 0$.

The $\mathcal{A}_{c,\tau}$ spaces are Banach spaces under a weighted Alexiewicz norm. A property that makes them somewhat delicate to work with is that they are not closed under translations.

Theorem 5.2. *Let $\tau > 0$.*

- (a) $\mathcal{B}_{c,\tau}$ is a Banach space isometrically isomorphic to \mathcal{B}_c .
- (b) $\mathcal{A}_{c,\tau}$ is a Banach space isometrically isomorphic to \mathcal{A}_c .
- (c) $f \in \mathcal{A}_{c,\tau}$ if and only if $f\omega_\tau \in \mathcal{A}_c$.
- (d) If $f \in \mathcal{A}_{c,\tau}$ then $\|f\|_\tau = \|f\omega_\tau\|$.
- (e) For each $\tau > 0$, $\mathcal{A}_c \subsetneq \mathcal{A}_{c,\tau}$.
- (f) $0 < r < s$ if and only if $\mathcal{A}_{c,s} \subsetneq \mathcal{A}_{c,r}$.
- (g) If $f \in \mathcal{A}_{c,\tau}$ and $F(x) = \int_0^x f$ then $|F(x)| \leq \|f\|_\tau e^{x^2/(4\tau)}$ for all $x \in \mathbb{R}$.
And, $F(x) = o(e^{x^2/(4\tau)})$ as $|x| \rightarrow \infty$.
- (h) If $0 < r < s$ then $\mathcal{A}_{c,s}$ is a dense subspace of $\mathcal{A}_{c,r}$.
- (i) If $0 < r < s$ then $\|f\|_r \leq 2\|f\|_s$ for each $f \in \mathcal{A}_{c,s}$.
- (j) Define $L^1(\omega_\tau) = \{f \in L^1_{loc} \mid f\omega_\tau \in L^1\}$. Then $L^1(\omega_\tau)$ is dense in $\mathcal{A}_{c,\tau}$.
- (k) $\mathcal{A}_{c,\tau}$ is not closed under translation.

Proof. (a) Using the uniqueness for Volterra integral equations, (5.1) defines a linear isometry between \mathcal{B}_c and $\mathcal{B}_{c,\tau}$. (b) Follows from the uniqueness of primitives proven above. (c) Let $F_\tau \in \mathcal{B}_{c,\tau}$. Then $F_\tau \in C(\mathbb{R})$ so F'_τ is integrable on compact intervals. From (5.1) let G be the corresponding function in \mathcal{B}_c . We have

$$G(x) - G(0) = F_\tau(x)\omega_\tau(x) + \frac{1}{2\tau} \int_0^x F_\tau(\xi)\xi\omega_\tau(\xi) d\xi = \int_0^x G' = \int_0^x F'_\tau\omega_\tau.$$

Therefore, $G' = F'_\tau\omega_\tau$ as elements in \mathcal{A}_c . The isometry between \mathcal{A}_c and $\mathcal{A}_{c,\tau}$ is then given by $\mathcal{A}_{c,\tau} \ni f \leftrightarrow f\omega_\tau \in \mathcal{A}_c$. If $f \in \mathcal{A}_{c,\tau}$ then $f\omega_\tau \in \mathcal{A}_c$ with primitive $F(x) = \int_{-\infty}^x f\omega_\tau$. The product $f\omega_\tau$ is well-defined via $\langle f\omega_\tau, \phi \rangle = \langle f, \phi\omega_\tau \rangle$ for each $\phi \in \mathcal{D}(\mathbb{R})$ since the integral $\int_{-\infty}^\infty (f\omega_\tau)\phi$ exists. (d)

$$\|f\|_\tau = \|F_\tau\|_{\tau,\infty} = \sup_{x < y} |G(x) - G(y)| = \sup_{x < y} \left| \int_x^y G' \right| = \sup_{x < y} \left| \int_x^y F'_\tau\omega_\tau \right| = \|f\omega_\tau\|.$$

(e) Since $\omega_\tau \in \mathcal{BV}$, if $f \in \mathcal{A}_c$ then $f\omega_\tau \in \mathcal{A}_c$ so $\mathcal{A}_c \subset \mathcal{A}_{c,\tau}$. (f) Suppose $0 < r < s$. Let $f \in \mathcal{A}_{c,s}$. Then $f\omega_s \in \mathcal{A}_c$. And, $f\omega_r = (f\omega_s)(\omega_r/\omega_s)$. The function $\omega_r/\omega_s \in \mathcal{BV}$ so $f \in \mathcal{A}_{c,r}$. The examples given below show the set inclusions in (e) and (f) are proper. Suppose $r > s > 0$. The function $f(x) = \exp(x^2/(4r))$ is in $\mathcal{A}_{c,s}$ but not in $\mathcal{A}_{c,r}$. The case $r = s$ is trivial. (g) Suppose $x > 0$. Write $\exp(-x^2/(4\tau))F(x) = \int_{-\infty}^\infty [f(\xi)\omega_\tau(\xi)]g_x(\xi) d\xi$ where $g_x(\xi) = e^{(\xi^2 - x^2)/(4\tau)}\chi_{[0,x]}(\xi)$. Note that $\lim_{x \rightarrow \infty} g_x(\xi) = 0$ for each $\xi \in \mathbb{R}$ and the variation of g_x on \mathbb{R} is 1. The Hölder inequality (2.3) gives the first estimate. By Theorem 2.1 we have $F(x) = o(\exp(x^2/(4\tau)))$ as $x \rightarrow \infty$. Similarly as $x \rightarrow -\infty$. (h) Given $f \in \mathcal{A}_{c,r}$ let $f_n = f\chi_{(-n,n)}$. Then $\{f_n\} \subset \mathcal{A}_{c,s}$. Let $\alpha < \beta$. Then $f\chi_{(\alpha,\beta)} \in \mathcal{A}_{c,r}$ and $f\omega_\tau\chi_{(\alpha,\beta)} \in \mathcal{A}_c$ so

$$\int_\alpha^\beta [f - f_n]\omega_r = \int_{-\infty}^{-n} f\omega_r\chi_{(\alpha,\beta)} + \int_n^\infty f\omega_r\chi_{(\alpha,\beta)}$$

and $\|f - f_n\|_r \leq \|f\omega_r\chi_{(-\infty, -n)}\| + \|f\omega_r\chi_{(n, \infty)}\| \rightarrow 0$ as $n \rightarrow \infty$. (i) The Hölder inequality (2.3) gives

$$\left| \int_{\alpha}^{\beta} f\omega_r \right| = \left| \int_{\alpha}^{\beta} [f\omega_s] \left[\frac{\omega_r}{\omega_s} \right] \right| \leq 2\|f\|_s.$$

(j) Let $\epsilon > 0$. Let $f \in \mathcal{A}_{c,\tau}$. Define $F(x) = \int_{-\infty}^x f\omega_r$. Then $F \in \mathcal{B}_c$. Note that the primitives of L^1 functions are given by $AC \cap \mathcal{BV}$. Since L^1 is dense in \mathcal{A}_c ([23, Proposition 3.3]) there is $G \in AC \cap \mathcal{BV}$ such that $\|F - G\|_{\infty} < \epsilon$. Now let $g = G'/\omega_r$. As $1/\omega_r$ is locally bounded we have $g \in L^1_{loc}$. Hence, $g \in L^1(\omega_r)$. And, $\|f - g\|_{\tau} \leq 2\|F - G\|_{\infty} < 2\epsilon$. (k) Let $f(x) = [(x^2 + 1)\omega_r(x)]^{-1}$. Then $f \in \mathcal{A}_{c,\tau}$. But $f(x+1)\omega_r(x) = \exp(x/[2\tau])\exp(1/[4\tau])[(x+1)^2 + 1]^{-1}$ so this translation is not in $\mathcal{A}_{c,\tau}$. \square

This shows that all of the spaces \mathcal{B}_c , \mathcal{A}_c , \mathcal{A}_c^n , $\mathcal{A}_{c,\tau}$, $\mathcal{B}_{c,\tau}$ are isometrically isomorphic.

If $\alpha > 0$ then $f(x) = \exp(x^2/(4\alpha)) \in \mathcal{A}_{c,\tau}$ if and only if $\alpha > \tau$. This shows that the distributions in $\mathcal{A}_{c,\tau}$ need not be tempered. And, $g(x) = \exp(\beta|x|^{\gamma}) \in \mathcal{A}_{c,\tau}$ for all $\tau > 0$ and all $\beta \in \mathbb{R}$ if and only if $0 \leq \gamma < 2$. For each $\tau > 0$ every polynomial is in $\mathcal{A}_{c,\tau}$. The Hölder inequality shows that $L^p \subset \mathcal{A}_{c,\tau}$ for each $1 \leq p \leq \infty$. Similarly for weighted L^p spaces. Let $\sigma > 0$ and $L^p(\omega_{\sigma})$ ($1 \leq p < \infty$) be the Lebesgue measurable functions for which $\int_{-\infty}^{\infty} |f(x)|^p \omega_{\sigma}(x) dx$ exists. The Hölder inequality shows $L^p(\omega_{\sigma}) \subset \mathcal{A}_{c,\tau}$ for all $\tau < p\sigma$ when $1 < p < \infty$ and $L^1(\omega_{\sigma}) \subset \mathcal{A}_{c,\tau}$ for all $\tau \leq \sigma$.

Now we look at analogues of Theorem 3.1 in weighted spaces. Proofs are similar to the corresponding parts of that theorem except as where noted.

Theorem 5.3. *Let $\tau > 0$, $0 < t < \tau$ and $f \in \mathcal{A}_{c,\tau}$. Let the primitive of f be $F \in C(\mathbb{R})$.*

- (a) *The integrals $f * \Theta_t(x) = \Theta_t * f(x) = F * \Theta'_t(x) = [F * \Theta_t]'(x)$ exist for each $x \in \mathbb{R}$ and $0 < t < \tau$.*
- (b) *$f * \Theta_t(x)$ is C^{∞} for $(x, t) \in \mathbb{R} \times (0, \tau)$.*
- (c) *Let $0 < t < \sigma \leq \tau$. For each $x \in \mathbb{R}$, $|f * \Theta_t(x)| \leq \|f\|_{\sigma} \exp(x^2/[4(\sigma - t)])/(2\sqrt{\pi t})$. Let $0 < t \leq \tau - \sigma$. The estimate $\|(f * \Theta_t)\omega_{\sigma}\|_{\infty} \leq \|f\|_{\tau}/(2\sqrt{\pi t})$ is sharp in the sense that the coefficient of $\|f\|_{\tau}$ cannot be reduced.*
- (d) *Let $\sigma \geq \tau$ and let $C(\omega_{\sigma}, \mathbb{R})$ be the Banach space of continuous functions with the norm $\|F\omega_{\sigma}\|_{\infty}$. Define the linear operator $\Phi_t: \mathcal{A}_{c,\tau} \rightarrow C(\omega_{\sigma}, \mathbb{R})$ by $\Phi_t(f) = f * \Theta_t$. Then $\|\Phi_t\| = 1/(2\sqrt{\pi t})$.*
- (e) *$f * \Theta_t(x) = o(\exp(x^2/(4[\tau - t])))$ as $|x| \rightarrow \infty$.*
- (f) *Let $0 < \sigma < \tau$ and $0 < t < \tau - \sigma$. Then $f * \Theta_t \in \mathcal{A}_{c,\sigma}$ and $\|f * \Theta_t\|_{\sigma} \leq \sqrt{(\tau - \sigma)/(\tau - \sigma - t)}\|f\|_{\sigma}$. Define the linear operator $\Psi_t: \mathcal{A}_{c,\sigma} \rightarrow \mathcal{A}_{c,\sigma}$ by $\Psi_t(f) = f * \Theta_t$. Then $\|\Psi_t\| = \sqrt{(\tau - \sigma)/(\tau - \sigma - t)}$.*
- (g) *Let $u(x, t) = f * \Theta_t(x)$ then $u \in C^2(\mathbb{R}) \times C^1((0, \tau))$ and u is a solution of the heat equation in this region. Let $0 < \sigma < \tau$. Then $\lim_{t \rightarrow 0^+} \|u_t - f\|_{\sigma} = 0$. (h) Let $0 < \sigma < \tau$ and $0 < t < \tau - \sigma$. Then $\int_{-\infty}^{\infty} f * \Theta_t(x)\Theta_{\sigma}(x) dx = \int_{-\infty}^{\infty} f(x)\Theta_{\sigma+t}(x) dx$.*

- (i) Let $0 < t < \tau - \sigma$. Then $\|f * \Theta_t\|_\sigma \leq \sqrt{\sigma(\tau - t)/[t(\tau - \sigma - t)]} \|f\|_\tau$. Now consider $\Psi_t : \mathcal{A}_{c,\tau} \rightarrow \mathcal{A}_{c,\sigma}$, as the restriction of the operator defined in part (f) to the subspace $\mathcal{A}_{c,\tau}$. Then $\|\Psi_t\| \leq \sqrt{\sigma(\tau - t)/[t(\tau - \sigma - t)]}$.
- (j) For each $0 < t < \tau$, $f * \theta_t(x)$ is real analytic as a function of $x \in \mathbb{R}$. For each $x \in \mathbb{R}$, $f * \Theta_t(x)$ is real analytic as a function of $0 < t < \tau$.

Proof. (a) This follows from the fact that for each $x \in \mathbb{R}$ and each $0 < t < \tau$ the function $\xi \mapsto \Theta_t(x - \xi)/\omega_\tau \in \mathcal{BV}$. The estimate in Theorem 5.2(g) allows integration by parts. Taylor's theorem shows that $[F * \Theta_t]'(x) = F' * \Theta_t(x)$. (b) Similar to Theorem 3.1(b). (c) Write

$$f * \Theta_t(x) = \int_{-\infty}^{\infty} [f(\xi)\omega_\sigma(\xi)] [\Theta_t(x - \xi)/\omega_\sigma(\xi)] d\xi.$$

The function $\xi \mapsto \Theta_t(x - \xi)/\omega_\sigma(\xi)$ is increasing on $(-\infty, \sigma x/(\sigma - t)]$ and decreasing on $[\sigma x/(\sigma - t), \infty)$. Using the second mean value theorem for integrals as in Theorem 3.1(c) we have,

$$(5.2) \quad f * \Theta_t(x) = \frac{\Theta_t(x - \sigma x/(\sigma - t))}{\omega_\sigma(\sigma x/(\sigma - t))} \int_{x_1}^{x_2} f\omega_\sigma = \frac{e^{\frac{x^2}{4(\sigma-t)}}}{2\sqrt{\pi t}} \int_{x_1}^{x_2} f\omega_\sigma,$$

where now $x_1 \leq \sigma x/(\sigma - t)$ and $x_2 \geq \sigma x/(\sigma - t)$. This gives the first inequality. Also, $|f * \Theta_t(x)\omega_\sigma(x)| \leq \exp(x^2/[4(\tau - t)]) \exp(-x^2/[4\sigma]) \|f\|_\tau/(2\sqrt{\pi t})$ and the second inequality follows. Using (2.4)-(2.6), the example $f = \Theta_s$ and the limit $s \rightarrow 0^+$ shows the estimate is sharp. (d) This follows from (c). (e) Write

$$e^{-x^2/(4[\tau-t])} f * \Theta_t(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} [f(\xi)\omega_\tau(\xi)] \left[e^{-(x-\xi)^2/(4t)} e^{\xi^2/(4\tau)} e^{-x^2/(4[\tau-t])} \right] d\xi.$$

The function $\xi \mapsto \exp(-(x - \xi)^2/(4t)) \exp(\xi^2/(4\tau)) \exp(-x^2/(4[\tau - t]))$ has variation as a function of ξ equal to 2. It has limit 0 as $|x| \rightarrow \infty$. The result follows by Theorem 2.1. (f) Let $0 < t < \tau - \sigma$ and $-\infty < \alpha < \beta < \infty$. From (a)

$$(5.3) \quad \int_{\alpha}^{\beta} f * \Theta_t(x)\omega_\sigma(x) dx = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f(x - \xi)\Theta_t(\xi)\omega_\sigma(x) d\xi dx$$

$$(5.4) \quad = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f(\xi)\Theta_t(x - \xi)\omega_\sigma(x) d\xi dx$$

$$(5.5) \quad = \int_{-\infty}^{\infty} [f(\xi)\omega_\tau(\xi)] \int_{\alpha}^{\beta} \frac{\Theta_t(x - \xi)\omega_\sigma(x)}{\omega_\tau(\xi)} dx d\xi.$$

We can interchange orders of integration in (5.5) since the function $\xi \mapsto \Theta_t(x - \xi)\omega_\sigma(x)/\omega_\tau(\xi)$ has variation $\exp(x^2/[4(\tau - t)]) \exp(-x^2/[4\sigma])/\sqrt{\pi t}$. The conditions of [23, Proposition A.3] are then satisfied. Now consider

$$(5.6) \quad \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} f(x - \xi)\Theta_t(\xi)\omega_\sigma(x) dx d\xi$$

$$(5.7) \quad = \int_{-\infty}^{\infty} \Theta_t(\xi) \int_{-\infty}^{\infty} [f(x)\omega_\tau(x)] \left[\frac{\omega_\sigma(x + \xi)}{\omega_\tau(x)} \right] \chi_{(\alpha-\xi, \beta-\xi)}(x) dx d\xi.$$

The change of variables is justified by [22, Theorem 11]. Note that $V_{x \in \mathbb{R}}[\omega_\sigma(x + \xi)/\omega_\tau(x)] = \exp(\xi^2/[4(\tau - \sigma)])$. Use the inequality $V(gh) \leq Vg\|h\|_\infty + \|g\|_\infty Vh$ with $g(x) = \omega_\sigma(x + \xi)/\omega_\tau(x)$ and $h(x) = \chi_{(\alpha - \xi, \beta - \xi)}(x)$ to see that the conditions of [23, Proposition A.3] are satisfied. We then have equality between (5.3) and (5.7), upon interchanging orders of integration in (5.7) and changing variables. This gives

$$\int_\alpha^\beta f * \Theta_t(x) \omega_\sigma(x) dx = \int_{-\infty}^\infty \Theta_t(\xi) \int_{-\infty}^\infty [f(x - \xi) \omega_\sigma(x - \xi) \chi_{(\alpha, \beta)}(x)] \psi(x) dx d\xi,$$

where $\psi(x) = \omega_\sigma(x)/\omega_\tau(x - \xi)$. Note that ψ has a maximum at $-\sigma\xi/(\tau - \sigma)$. Using the second mean value theorem as in Theorem 3.1(c), there are $x_1 \leq -\sigma\xi/(\tau - \sigma) \leq x_2$ such that

$$\begin{aligned} (5.8) \quad & \left| \int_\alpha^\beta f * \Theta_t(x) \omega_\sigma(x) dx \right| \\ &= \left| \int_{-\infty}^\infty \Theta_t(\xi) e^{\xi^2/[4(\tau - \sigma)]} \int_{(x_1, x_2) \cap (\alpha, \beta)} f(x - \xi) \omega_\sigma(x - \xi) dx d\xi \right| \\ &\leq \|f\|_\sigma \int_{-\infty}^\infty \Theta_t(\xi) e^{\xi^2/[4(\tau - \sigma)]} d\xi = \|f\|_\sigma \sqrt{\frac{\tau - \sigma}{\tau - \sigma - t}}. \end{aligned}$$

This also shows the limits exist as $\alpha \rightarrow -\infty$ and $\beta \rightarrow \infty$. Hence, $f * \Theta_t \in \mathcal{A}_{c, \sigma}$. If we let $f = \Theta_{-s}$ for $s > \tau$ we get $f * \Theta_t = \Theta_{t-s}$, $\|f\|_\sigma = \sqrt{\sigma/(s - \sigma)}$ and $\|f * \Theta_t\|_\sigma = \sqrt{\sigma/(s - \sigma - t)}$. Letting $s \rightarrow \tau^+$ shows the estimate is sharp and gives the norm of Ψ_t . (g) Since we can differentiate under the integral sign, u is a solution of the heat equation in $\mathbb{R} \times (0, \tau)$. To show the initial conditions are taken on in the weighted norm, use the equality between (5.3) and (5.7). This gives

$$\int_\alpha^\beta [f * \Theta_t(x) - f(x)] \omega_\sigma(x) dx = \int_{-\infty}^\infty \Theta_t(\xi) \int_\alpha^\beta [f(x - \xi) - f(x)] \omega_\sigma(x) dx d\xi.$$

Write

$$\int_\alpha^\beta [f(x - \xi) - f(x)] \omega_\sigma(x) dx = I_1(\xi) - I_2(\xi) + I_3(\xi)$$

where

$$\begin{aligned} I_1(\xi) &= \int_{\alpha - \xi}^\alpha f(x) \omega_\sigma(x + \xi) dx \\ I_2(\xi) &= \int_{\beta - \xi}^\beta f(x) \omega_\sigma(x + \xi) dx \\ I_3(\xi) &= \int_\alpha^\beta f(x) [\omega_\sigma(x + \xi) - \omega_\sigma(x)] dx. \end{aligned}$$

Now show integrals of I_1, I_2, I_3 against $\Theta_t(\xi)$ tend to 0 as $t \rightarrow 0^+$, uniformly in α and β . Let $F_\tau(x) = \int_{-\infty}^x f \omega_\tau$. Use the Hölder inequality (2.3) to get

$$\begin{aligned} \int_{-\infty}^{\infty} \Theta_t(\xi) |I_1|(\xi) d\xi &\leq 2 \int_{-\infty}^{\infty} \Theta_t(\xi) \sup_{\alpha \in \mathbb{R}} \max_{y, z \in [\alpha - \xi, \alpha]} |F_\tau(y) - F_\tau(z)| e^{\xi^2/[4(\tau - \sigma)]} d\xi \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0^+ \end{aligned}$$

since F_τ is uniformly continuous on \mathbb{R} and Θ_t is a summability kernel (approximate identity). Similarly, for $\xi < 0$. Similarly, for I_2 . And,

$$|I_3|(\xi) \leq 2 \|F_\tau\|_\infty \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} \frac{\omega_\sigma(x + \xi) - \omega_\sigma(x)}{\omega_\tau(x)} \right| dx.$$

Dominated convergence allows us to take the limit $\xi \rightarrow 0$ under the integral sign to get $I_3(\xi) \rightarrow 0$. Using (5.3), (5.7), (2.4) we have

$$\|f * \Theta_t - f\|_\sigma \leq \int_{-\infty}^{\infty} \Theta_t(\xi) \|f(\cdot - \xi) - f(\cdot)\|_\sigma d\xi$$

and $\|f(\cdot - \xi) - f(\cdot)\|_\sigma$ is continuous at $\xi = 0$. But, Θ_t is a summability kernel so $\lim_{t \rightarrow 0^+} \|u_t - f\|_\sigma = 0$. (h) The calculation following (5.5) shows that

$$\int_{-\infty}^{\infty} f * \Theta_t(x) \Theta_\sigma(x) dx = \int_{-\infty}^{\infty} f(\xi) \Theta_t * \Theta_\sigma(\xi) d\xi = \int_{-\infty}^{\infty} f(\xi) \Theta_{\sigma+t}(\xi) d\xi.$$

(i) We can write

$$\int_{\alpha}^{\beta} f * \Theta_t(x) \omega_\sigma(x) dx = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} [f(\xi) \omega_\tau(\xi)] \left[\frac{\Theta_t(x - \xi)}{\omega_\tau(\xi)} \right] d\xi \omega_\sigma(x) dx.$$

Now use (5.2) to get

$$\|f * \Theta_t\|_\sigma \leq \frac{2\sqrt{\pi\sigma}\sqrt{\tau-t}\Theta_{t-\tau} * \Theta_\sigma(0)\|f\|_\tau}{\sqrt{t}} = \sqrt{\frac{\sigma(\tau-t)}{t(\tau-\sigma-t)}} \|f\|_\tau.$$

(j) See Theorem 10.2.1, Theorem 10.3.1 in [5] and the proof of Theorem 3.1(j). Part (g) of Theorem 5.2 gives the necessary growth condition on F . \square

Note that in (f) the coefficient remains bounded as $t \rightarrow 0^+$ but not so in (i). This will be important for uniqueness Theorem 6.6 below.

Examples 5.4. (a) If $f \in \mathcal{A}_{c,\tau}$ then $f * \Theta_t$ need not be in $\mathcal{A}_{c,\tau}$. Let $f(x) = 1/[(x^2 + 1)\omega_\tau(x)]$. Since $f \geq 0$, for each $t > 0$ the Fubini–Tonelli theorem gives

$$\begin{aligned} \|f * \Theta_t\|_\tau &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Theta_t(x - \xi) \omega_\tau(x)}{\omega_\tau(\xi)(\xi^2 + 1)} d\xi dx = 2\sqrt{\pi\tau} \int_{-\infty}^{\infty} \frac{e^{\xi^2/(4\tau)}}{\xi^2 + 1} \Theta_t * \Theta_\tau(\xi) d\xi \\ &= 2\sqrt{\pi\tau} \int_{-\infty}^{\infty} \frac{e^{\xi^2/(4\tau)} \Theta_{t+\tau}(\xi)}{\xi^2 + 1} d\xi = \infty. \end{aligned}$$

(b) Let $s > \tau$ and take $f = \Theta_{-s}$. Then $f \in \mathcal{A}_{c,\tau}$ and for $0 < t < s$ formula (2.5) gives $f * \Theta_t(x) = \Theta_{t-s}(x) = \exp(x^2/[4(s-t)])/[2\sqrt{\pi(s-t)}]$. Note that if $0 < \sigma < s$ then $f * \Theta_t \in \mathcal{A}_{c,\sigma}$ if and only if $0 < t < s - \sigma$. In particular, $f * \Theta_t \in \mathcal{A}_{c,\tau}$ if and only if $0 < t < s - \tau$.

(c) Let z be a complex number. If $f(x) = \exp(izx)$ then we get the plane wave solution $f * \Theta_t(x) = e^{izx - z^2 t}$ (c.f. [16, p. 207]). For all complex z , both f and $f * \Theta_t$ are in $\mathcal{A}_{c,\tau}$ for all $\tau > 0$.

(d) If f is a polynomial of degree n then $f * \Theta_t$ is a polynomial in x and t of degree n in x and degree $\lfloor n/2 \rfloor$ in t . To see this let $p_n(x) = x^n$. The binomial theorem followed by basic gamma function identities then give

$$\begin{aligned} p_n * \Theta_t(x) &= \frac{1}{\sqrt{\pi}} \sum_{m=0}^n (-1)^m \binom{n}{m} x^{n-m} (2\sqrt{t})^m \int_{-\infty}^{\infty} \xi^m e^{-\xi^2} d\xi \\ &= \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{n! x^{n-2\ell} t^\ell}{(n-2\ell)! \ell!} = (i\sqrt{t})^n H_n(x/[2i\sqrt{t}]). \end{aligned}$$

The last line comes from the explicit form of the Hermite polynomial. See [11, 8.950.2]. For example, $p_3 * \Theta_t(x) = x^3 + 6xt$. In the literature these are called heat polynomials [29]. For each polynomial f we have $f * \Theta_t \in \mathcal{A}_{c,\tau}$ for each $\tau > 0$. This also gives $H_n * \Theta_t(x) = (1-4t)^{n/2} H_n(x/\sqrt{1-4t})$. The above quoted formula shows this is valid for all $t > 0$.

(e) If we take $g(x) = \partial \exp(ix^2/(4s))/\partial s = -ix^2 \exp(ix^2/(4s))/(4s^2)$. Then we can get $g * \Theta_t$ by differentiating the solution in (3.3). Note that $g \in \mathcal{A}_{c,\tau}$ for each $\tau > 0$.

(f) Let p be a polynomial, $a > 0$, $b \in \mathbb{R}$, $0 < \gamma < 1$. Define $f(x) = p(x) \exp(x^2/(4a) + b|x|^\gamma)$. Let $0 < \tau < a$. Writing $G(x) = \int_{-\infty}^x f(\xi) \omega_\tau(\xi) d\xi$ defines $G \in \mathcal{B}_c$. Since $G' = f \omega_\tau$ we have $f \in \mathcal{A}_{c,\tau}$.

Proposition 5.5. *Let $0 < \sigma < \tau$ and $0 < t < \tau - \sigma$. Let $f \in \mathcal{A}_{c,\tau}$ and $u(x, t) = f * \Theta_t(x)$. Then $\|u'_t\|_\sigma \leq (\tau - \sigma)/[\sqrt{\pi t}(\tau - \sigma - t)]$;*

$$\|\partial u(\cdot, t)/\partial t\|_\sigma = \|u''_t\|_\sigma \leq \frac{\|f\|_\sigma}{2t} \left[\frac{(\tau - \sigma)^{3/2}}{(\tau - \sigma - t)^{3/2}} + \frac{\sqrt{\tau - \sigma}}{\sqrt{\tau - \sigma - t}} \right];$$

$$|u'_t(x)| \leq \frac{e^{x^2/4(\sigma-t)} \|f\|_\sigma \sqrt{\sigma}}{\sqrt{\sigma-t}} \left[\frac{1}{t} + \frac{|x|}{2\sqrt{\pi\sigma(\sigma-t)t}} \right].$$

Proof. As in (5.8),

$$\left| \int_\alpha^\beta f * \Theta'_t(x) \omega_\sigma(x) dx \right| \leq \|f\|_\sigma \int_{-\infty}^{\infty} |\Theta'_t(\xi)| e^{\xi^2/[4(\tau-\sigma)]} d\xi = \frac{(\tau - \sigma) \|f\|_\sigma}{\sqrt{\pi t}(\tau - \sigma - t)}.$$

Similarly,

$$\begin{aligned} \left| \int_\alpha^\beta f * \Theta''_t(x) \omega_\sigma(x) dx \right| &\leq \|f\|_\sigma \int_{-\infty}^{\infty} |\Theta''_t(\xi)| e^{\xi^2/[4(\tau-\sigma)]} d\xi \\ &\leq \frac{\|f\|_\sigma}{2t} \int_{-\infty}^{\infty} \Theta_t(\xi) e^{\xi^2/[4(\tau-\sigma)]} \left(\frac{\xi^2}{2t} + 1 \right) d\xi \\ &= \frac{\|f\|_\sigma}{2t} \left[\frac{(\tau - \sigma)^{3/2}}{(\tau - \sigma - t)^{3/2}} + \frac{\sqrt{\tau - \sigma}}{\sqrt{\tau - \sigma - t}} \right]. \end{aligned}$$

As in the proof of Theorem 5.3(c),

$$|u'_t(x)| \leq \|f\|_\sigma V_{\xi \in \mathbb{R}} \frac{\Theta'_t(x - \xi)}{\omega_\sigma(\xi)}.$$

Note that

$$\begin{aligned} V_{\xi \in \mathbb{R}} \frac{\Theta'_t(x - \xi)}{\omega_\sigma(\xi)} &= \frac{1}{4\sqrt{\pi} t^{3/2}} \int_{-\infty}^{\infty} e^{-(\xi-x)^2/(4t)} e^{\xi^2/(4\sigma)} \left| \frac{(\xi-x)^2}{2t} - 1 - \frac{\xi(\xi-x)}{2\sigma} \right| d\xi \\ &= \frac{e^{x^2/(4(\sigma-t))} \sqrt{\sigma}}{2\sqrt{\pi(\sigma-t)} t} \int_{-\infty}^{\infty} e^{-s^2} \left| 2s^2 + xs \sqrt{\frac{t}{\sigma(\sigma-t)}} - 1 \right| ds, \end{aligned}$$

upon completing the square. \square

An inequality for $|\partial u(x, t)/\partial t| = |u''_t(x)|$ can be proved in the same manner.

6. UNIQUENESS

The example $u(x, t) = \delta' * \Theta_t(x) = \Theta'_t(x) = -x \exp(-x^2/(4t))/(4\sqrt{\pi} t^{3/2})$ shows that the heat equation need not have a unique solution, since $u(x, 0) = 0$ for all x . Uniqueness can be obtained by imposing a boundedness condition on norms of solutions. We prove uniqueness by reducing to one of the two classical theorems.

Theorem 6.1. *Let $u \in C^2(\mathbb{R}) \times C^1((0, \infty))$ such that $u \in C(\mathbb{R} \times [0, \infty))$, $u_2 - u_{11} = 0$ in $\mathbb{R} \times (0, \infty)$, u is bounded, $u(x, 0) = f(x)$ for a bounded continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$. Then the unique solution is given by $u_t(x) = f * \Theta_t(x)$.*

Theorem 6.2. *Let $\tau > 0$. Let $u \in C^2(\mathbb{R}) \times C^1((0, \tau))$ such that $u \in C(\mathbb{R} \times [0, \tau))$, $u_2 - u_{11} = 0$ in $\mathbb{R} \times (0, \tau)$, $|u(x, t)| \leq Ae^{Bx^2}$ for some constants A, B and all $(x, t) \in \mathbb{R} \times [0, \tau)$, $u(x, 0) = f(x)$ for a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $|f(x)| \leq Ae^{Bx^2}$. Then the unique solution is given by $u_t(x) = f * \Theta_t(x)$.*

The second theorem is due to Tychonoff. Widder provides a proof in [28] and also discusses various related results: Täcklind's generalisation of the allowed exponent and a uniqueness condition for non-negative solutions. See also [29].

For uniqueness in \mathcal{A}_c , \mathcal{A}_c^n , or $\mathcal{A}_{c,\tau}$, pointwise conditions are not applicable. However, we obtain a uniqueness condition by requiring boundedness of the solution in the Alexiewicz norm. We use a method similar to that used by Hirschman and Widder in proving uniqueness under an L^p norm condition [14, Theorem 9.2].

Theorem 6.3 (Uniqueness in \mathcal{A}_c). *Let $u \in C^2(\mathbb{R}) \times C^1((0, \infty))$ such that $u_2 - u_{11} = 0$ in $\mathbb{R} \times (0, \infty)$, $\|u_t\|$ is bounded, $\lim_{t \rightarrow 0^+} \|u_t - f\| = 0$ for some $f \in \mathcal{A}_c$. The unique solution is then given by $u_t(x) = f * \Theta_t(x)$.*

Proof. From Theorem 3.1 we see that $f * \Theta_t$ satisfies the hypotheses.

If u and v are two solutions, let $h = u - v$. Then $h \in C^2(\mathbb{R}) \times C^1((0, \infty))$, $h_2 - h_{11} = 0$ in $\mathbb{R} \times (0, \infty)$, $\|h_t\|$ is bounded, $\lim_{t \rightarrow 0^+} \|h_t\| = 0$. Let $y, t > 0$ and

let $\psi_y(x, t) = (2y)^{-1} \int_{x-y}^{x+y} h_t(\xi) d\xi$. Then

$$\frac{\partial \psi_y(x, t)}{\partial t} = \frac{1}{2y} \int_{x-y}^{x+y} \frac{\partial h(\xi, t)}{\partial t} d\xi = \frac{1}{2y} \int_{x-y}^{x+y} h_t''(\xi) d\xi = \frac{h_t'(x+y) - h_t'(x-y)}{2y}.$$

And,

$$\frac{\partial^2 \psi_y(x, t)}{\partial x^2} = \frac{\partial}{\partial x} \frac{h(x+y, t) - h(x-y, t)}{2y} = \frac{h_1(x+y, t) - h_1(x-y, t)}{2y}.$$

Hence, $\psi_y \in C^2(\mathbb{R}) \times C^1((0, \infty))$ and is a solution of the heat equation. Note that $|\psi_y(x, t)| \leq \|h_t\|/(2y)$. It then follows that ψ_y is bounded for $(x, t) \in \mathbb{R} \times (0, \infty)$ and tends to 0 as $t \rightarrow 0^+$. Define $\psi_y(x, 0) = 0$. Let $x \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{R} \times (0, \infty)$. Then $|\psi_y(x, 0) - \psi_y(\alpha, \beta)| \leq \|h_\beta\|/(2y) \rightarrow 0$ as $(\alpha, \beta) \rightarrow (x, 0)$. Hence, $\psi_y \in C(\mathbb{R} \times [0, \infty))$. By Theorem 6.1 we have $\psi_y = 0$ for each $y > 0$. By the continuity of h we get $\lim_{y \rightarrow 0^+} \psi_y(x, t) = 0 = h(x, t)$. \square

Lemma 6.4. *Let $n \in \mathbb{N}$. Let $g_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k \chi_{(a_k, a_{k+1})}(x)$, where $a_k = k/(n+1)$. Define $G_n(x) = \int_0^x \cdots \int_0^{x_{i+1}} \cdots \int_0^{x_2} g_n(x_1) dx_1 \cdots dx_i \cdots dx_n$. Then $G_n \in C^{n-1}([0, 1])$, $G_n(x) = O(x^n)$ as $x \rightarrow 0^+$, $G_n(x) = O((x-1)^n)$ as $x \rightarrow 1^-$ and $G_n > 0$ on $(0, 1)$.*

Proof. The definition shows $G_n \in C^{n-1}([0, 1])$.

We have $G_n(x) = [(n-1)!]^{-1} \int_0^x (x-x_1)^{n-1} g_n(x_1) dx_1$. To prove that $G_n > 0$ it suffices to evaluate at each a_k . Hence,

$$\begin{aligned} G_n(a_{l+1}) &= \frac{1}{(n-1)!} \sum_{k=0}^l \binom{n}{k} (-1)^k \int_{a_k}^{a_{k+1}} (a_{l+1} - x_1)^{n-1} dx_1 \\ &= \frac{1}{n!(n+1)^n} \sum_{k=0}^l \binom{n+1}{k} (-1)^k (l+1-k)^n \\ &= \frac{A(n, l)}{n!(n+1)^n}, \end{aligned}$$

where $A(n, l)$ is the Eulerian number of the first kind. Combinatoric arguments show these are positive for $0 \leq l < n$ and that $A(n, n) = 0$. See, for example, [12].

It is clear that

$$\lim_{x \rightarrow 0^+} \frac{G_n(x)}{x^n} = \lim_{x \rightarrow 0^+} \frac{G_n^{(n)}(x)}{n!} = \frac{g_n(0^+)}{n!} = \frac{1}{n!}.$$

Note that $1-y \in (a_k, a_{k+1})$ if and only if $y \in (a_{n-k}, a_{n-k+1})$. So, $g_n(1-y) = (-1)^n g_n(y)$. A change of variables then shows

$$(6.1) \quad G_n(1-x) = G_n(x) + \frac{(-1)^n}{(n-1)!} \int_0^1 (y-x)^{n-1} g_n(y) dy.$$

And,

$$\begin{aligned}
 \int_0^1 (y-x)^{n-1} g_n(y) dy &= \sum_{k=0}^n \binom{n}{k} (-1)^k \int_{a_k}^{a_{k+1}} (y-x)^{n-1} dy \\
 &= \frac{(-1)^{n+1}}{n(n+1)^n} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k [(n+1)x - k]^n \\
 (6.2) \quad &= -\frac{1}{n} \sum_{m=0}^n \binom{n}{m} \frac{(-x)^{n-m}}{(n+1)^m} T_{n+1,m},
 \end{aligned}$$

where $T_{n,m} = \sum_{k=0}^n \binom{n}{k} (-1)^k k^m$. (The term k^m is defined to be 1 when $k = m = 0$.) Let $S_n(x) = (1-x)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k x^k$. Then

$$S_n^{(m)}(1) = 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k k(k-1) \cdots (k-m+1)$$

for $0 \leq m \leq n-1$. It follows that $T_{n,m}$ is a linear combination of $S_n^{(p)}(1)$ for $0 \leq p \leq m$. Hence, $T_{n,m} = 0$ ($0 \leq m \leq n-1$). From (6.1) and (6.2) we see that $G_n(1-x) = G_n(x)$. This gives the order relation as $x \rightarrow 1^-$. \square

Theorem 6.5 (Uniqueness in \mathcal{A}_c^n). *Let $n \geq 2$. Let $u \in C^2(\mathbb{R}) \times C^1((0, \infty))$ such that $u_2 - u_{11} = 0$ in $\mathbb{R} \times (0, \infty)$, $\|u_t\|^{(n)}$ is bounded, $\lim_{t \rightarrow 0^+} \|u_t - f\|^{(n)} = 0$ for some $f \in \mathcal{A}_c^n$. The unique solution is then given by $u_t(x) = f * \Theta_t(x)$.*

Proof. Theorem 4.6(a),(b),(f),(g) show that $f * \Theta_t$ satisfies the hypotheses. The other part of the proof is similar to that of Theorem 6.3. Use the same notation for h_t . From Lemma 6.4, the function $\xi \mapsto G_{n-1}((\xi - x + y)/(2y)) \chi_{(x-y, x+y)}(\xi)$ is in \mathcal{IBV}^{n-1} (Definition 4.1). Since $h_t \in \mathcal{A}_c^n$ the integral

$$\begin{aligned}
 \psi_y(x, t) &= \frac{1}{2y} \int_{-\infty}^{\infty} G_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) \chi_{(x-y, x+y)}(\xi) h_t(\xi) d\xi \\
 &= \frac{1}{2y} \int_{x-y}^{x+y} G_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) h_t(\xi) d\xi
 \end{aligned}$$

exists. Integrating by parts gives

$$\begin{aligned}
 \frac{\partial \psi_y(x, t)}{\partial t} &= \frac{1}{2y} \int_{x-y}^{x+y} G_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) \frac{\partial h(\xi, t)}{\partial t} d\xi \\
 &= -\frac{1}{(2y)^2} \int_{x-y}^{x+y} G'_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) h'_t(\xi) d\xi.
 \end{aligned}$$

And,

$$\begin{aligned}
 \frac{\partial^2 \psi_y(x, t)}{\partial x^2} &= \frac{1}{(2y)^3} \int_{x-y}^{x+y} G''_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) h_t(\xi) d\xi \\
 &= -\frac{1}{(2y)^2} \int_{x-y}^{x+y} G'_{n-1} \left(\frac{\xi}{2y} - \frac{x-y}{2y} \right) h'_t(\xi) d\xi,
 \end{aligned}$$

so that ψ_y is a solution of the heat equation. From Proposition 4.4,

$$|\psi_y(x, t)| \leq \frac{\|h_t\|^{(n)}}{(2y)^n} \left[\inf_{[0,1]} |g_{n-1}| + V_{[0,1]} g_{n-1} \right] = \frac{\|h_t\|^{(n)}}{y^n} \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

For fixed $y > 0$, ψ_y can then be continuously extended to vanish on $t = 0$. Theorem 6.1 shows $\psi_y = 0$. Let $(x, t) \in \mathbb{R} \times (0, \infty)$. Since h_t is continuous and $G_{n-1} > 0$ in $(0, 1)$ the mean value theorem for integrals shows there is $x^* \in (x - y, x + y)$ such that

$$\lim_{y \rightarrow 0^+} \psi_y(x, t) = 0 = \lim_{y \rightarrow 0^+} h_t(x^*) \int_0^1 G_{n-1}(\xi) d\xi = h_t(x) \int_0^1 G_{n-1}(\xi) d\xi.$$

It now follows that $h_t(x) = 0$. □

Theorem 6.6 (Uniqueness in $\mathcal{A}_{c,\tau}$). *Let $\tau > 0$. Let $u \in C^2(\mathbb{R}) \times C^1((0, \tau))$ such that $u_2 - u_{11} = 0$ in $\mathbb{R} \times (0, \tau)$. Suppose that for each $0 < \sigma < \tau$ the quantity $\|u_t\|_\sigma \sqrt{\tau - \sigma - t}$ is bounded for all $0 < t < \tau - \sigma$ and $\lim_{t \rightarrow 0^+} \|u_t - f\|_\sigma = 0$ for some $f \in \mathcal{A}_{c,\tau}$. The unique solution is then given by $u_t(x) = f * \Theta_t(x)$.*

Proof. Theorem 5.3(a),(b),(f),(g) show that $f * \Theta_t$ satisfies the hypotheses. The other part of the proof is similar to that of Theorem 6.3. Use the same notation. As before, h and ψ_y are solutions of the heat equation in $\mathbb{R} \times (0, \tau)$. Now use the Hölder inequality (2.3) to get

$$|\psi_y(x, t)| = \frac{1}{2y} \left| \int_{x-y}^{x+y} [h_t(\xi) \omega_\sigma(\xi)] \frac{d\xi}{\omega_\sigma(\xi)} \right| \leq \frac{\|h_t\|_\sigma}{y} \left[e^{(x+y)^2/(4\sigma)} + e^{(x-y)^2/(4\sigma)} \right].$$

If $0 < y \leq 1$ then $|\psi_y(x, t)| \leq 2\|h_t\|_\sigma \exp(1/\sigma) \exp(x^2/\sigma)/y$. Fix $0 < \rho < \tau$ and let $0 < \sigma < \rho$. We then have $A = 2 \sup_{0 < t \leq \tau - \rho} \|h_t\|_\sigma \exp(1/\sigma)/y$ and $B = 1/\sigma$ in Theorem 6.2. Define $\psi_y(x, 0) = 0$. Let $x \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{R} \times (0, \tau)$. Then $|\psi_y(x, 0) - \psi_y(\alpha, \beta)| \leq 2\|h_\beta\|_\sigma \exp(1/\sigma) \exp(\alpha^2/\sigma)/y \rightarrow 0$ as $(\alpha, \beta) \rightarrow (x, 0)$. Hence, $\psi_y \in C(\mathbb{R} \times [0, \tau))$. By Theorem 6.2 we have $\psi_y = 0$ on $\mathbb{R} \times [0, \tau - \rho)$ for each $y > 0$. By the continuity of h we get $\lim_{y \rightarrow 0^+} \psi_y(x, t) = 0 = h(x, t)$ for each $(x, t) \in \mathbb{R} \times (0, \tau - \rho)$. But, $0 < \rho < \sigma < \tau$ were arbitrary so we have $h = 0$ on $\mathbb{R} \times (0, \tau)$. □

REFERENCES

- [1] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, Colloquium Math. **1**(1948), 289–293.
- [2] D.D. Ang, K. Schmitt and L.K. Vy, *A multidimensional analogue of the Denjoy–Perron–Henstock–Kurzweil integral*, Bull. Belg. Math. Soc. Simon Stevin **4**(1997), 355–371.
- [3] B. Bäumer, G. Lumer and F. Neubrander, *Convolution kernels and generalized functions*, in: I. Antoniou and G. Lumer (Eds.), Generalized functions, operator theory, and dynamical systems, Boca Raton, Chapman and Hall/CRC, 1999, pp. 68–78.
- [4] B. Bongiorno and T.V. Panchapagesan, *On the Alexiewicz topology of the Denjoy space*, Real Anal. Exchange **21**(1995–1996), 604–614.
- [5] J.R. Cannon, *The one-dimensional heat equation*, Menlo Park, Addison–Wesley, 1984.
- [6] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology, vol. 5* (trans. A. Craig), Berlin, Springer–Verlag, 1992.
- [7] B. Epstein, *Partial differential equations*, New York, McGraw–Hill, 1962.
- [8] G.B. Folland, *Real analysis*, New York, Wiley, 1999.

- [9] F.G. Friedlander and M. Joshi, *Introduction to the theory of distributions*, Cambridge, Cambridge University Press, 1999.
- [10] R.A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Providence, American Mathematical Society, 1994.
- [11] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products* (trans. Scripta Technica, Inc., eds. A. Jeffrey and D. Zwillinger), San Diego, Academic Press, 2000.
- [12] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete mathematics*, Reading, Addison-Wesley, 1994.
- [13] K.E. Gustafson, *Introduction to partial differential equations and Hilbert space methods*, New York, Dover, 1999.
- [14] I.I. Hirschman and D.V. Widder, *The convolution transform*, Princeton, Princeton University Press, 1955.
- [15] R.J. Iorio, Jr. and V. de Magalhaes Iorio, *Fourier analysis and partial differential equations*, New York, Cambridge University Press, 2001.
- [16] F. John, *Partial differential equations*, New York, Springer-Verlag, 1982.
- [17] E.H. Lieb and M. Loss, *Analysis*, Providence, American Mathematical Society, 2001.
- [18] R.M. McLeod, *The generalized Riemann integral*, Washington, The Mathematical Association of America, 1980.
- [19] P. Mikusiński and K. Ostaszewski, *Embedding Henstock integrable functions into the space of Schwartz distributions*, Real Anal. Exchange **14**(1988-1989), 24–29.
- [20] Z. Szmydt, *Fourier transforms and linear differential equations* (trans. M. Kuczma), Dordrecht, D. Reidel, 1977.
- [21] E. Talvila, *Henstock-Kurzeil Fourier transforms*, Illinois J. Math. **46**(2002), 1207–1226.
- [22] E. Talvila, *The distributional Denjoy integral*, Real Anal. Exchange **33**(2008), 51–82.
- [23] E. Talvila, *Convolutions with the continuous primitive integral*, Abstr. Appl. Anal. **2009**(2009), Art. ID 307404. This version of the paper contains a number of errors inserted by the publisher that the publisher refused to correct. A corrected version can be found at <http://front.math.ucdavis.edu/>.
- [24] E. Talvila, *The regulated primitive integral*, Illinois J. Math. **53**(2009), 1187–1219.
- [25] E. Talvila, *Integrals and Banach spaces for finite order distributions*, Czechoslovak Math. J. **62(137)**(2012), 77–104.
- [26] E. Talvila, *The L^p primitive integral*, Math. Slovaca **64**(2014), 1497–1524.
- [27] N. Watson, *Introduction to heat potential theory*, Providence, American Mathematical Society, 2012.
- [28] D.V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. **55**(1944), 85–95.
- [29] D.V. Widder, *The heat equation*, New York, Academic Press, 1975.
- [30] K. Yosida, *Lectures on differential and integral equations*, New York, Interscience, 1960.

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF THE FRASER VALLEY,
 ABBOTSFORD, BC CANADA V2S 7M8

E-mail address: Erik.Talvila@ufv.ca